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ON THE EQUIVALENCE OF THE ZUBAREV METHOD AND THE BOHM-PINES METHOD FOR SYSTEMS OF TWO TYPES OF PARTICLES

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In this work, the method presented by Bohm and Pines (Phys. Rev., **92**, 609 (1953)) for the collective description of systems of two types of particles (electrons and ions) has been generalized. The result is compared with the results previously obtained by the author (Galasiewicz, Acta phys. Polon., **15**, 49 (1956)) for the generalization of the Zubarev method of supplementary variables (J. Phys. USSR, **25**, 548 (1953)) for the same case. The equivalence of both methods is shown. The unitary transformation which transforms the Hamiltonian obtained by one method into the Hamiltonian obtained by the other is given.

1. Introduction

In the papers of Zubarev (1953 a, b) and Bohm and Pines (1951, 1953), — Bohm and Pines will, hereafter, be referred to as BP — new methods for considering systems of many particles on the basis of quantum mechanics have been presented. Both methods are applied to a system of a great number of electrons (fermions). As a consequence of the use of these methods, the term representig the energy of the collective oscillations of the system, which follow Bose-Einstein statistics, appears as a separate term in the Hamiltonian.

In the Zubarev method, supplementary variables which are to describe the collective motion of the system are introduced into the wave function of the system, while a certain subsidiary condition is imposed on the wave function. The Hamiltonian of the system represents the energy of the electrons interacting with each other by means of a Coulomb potential.

The subsidiary condition and the Hamiltonian are transformed by a suitable unitary transformation. After making use of the subsidiary condition, the following terms appear in the Hamiltonian: 1) terms dependent only on the coordinates and momenta of the particles, 2) terms dependent only on the supplementary variables and their canonical conjugates, 3) terms dependent on the supplementary variables

and the coordinates of the particles. The terms in the second group represent the energy of the collective oscillations of the system.

The method of collective description takes as its starting point the formalism used in quantum electrodynamics with the longitudinal electric field eliminated (see, for example, Wentzel 1943, § 17). The Hamiltonian of the system represents the energy of the electrons moving in the longitudinal electric field and the energy of this field. We impose on the wave function a subsidiary condition defining the source of the field. The Hamiltonian and the subsidiary condition are transformed by a suitable unitary transformation. In the new Hamiltonian, the following terms appear: 1) terms dependent only on the coordinates and momenta of the particles, 2) terms dependent only on the „collective variables" (i. e., the Fourier components of the longitudinal electric field intensity and the vector potential of this field, 3) terms dependent on the coordinates of the particles and the collective variables.

In a preceding paper (Galasiewicz 1955), the method of supplementary variables for systems of two types of particles — electrons (fermions) and ions (fermions or bosons) — has been generalized. In the present work, the collective description method has been generalized for the same case. It is shown that both methods are equivalent.

2. Results obtained by generalizing the supplementary variables method

The Hamiltonian examined in the paper just mentioned (Galasiewicz 1955) had the form:

$$H = \sum_{j/1}^{N_1} \frac{p_j^2}{2m} + 2\pi e^2 \sum_{k, j_1, j_2} \frac{e^{ik(r_{j_1} - r_{j_2})}}{k^2} - 2\pi N_1 e^2 \sum_k \frac{1}{k^2} +$$

$$+ \sum_{l/1}^{N_2} \frac{\pi_l^2}{2M} + 2\pi Z^2 e^2 \sum_{k, l_1, l_2} \frac{e^{ik(R_{l_1} - R_{l_2})}}{k^2} - 2\pi N_2 Z^2 e^2 \sum_k \frac{1}{k^2} + \qquad (1)$$

$$- 4\pi Z e^2 \sum_{k, j, l} \frac{e^{ik(r_j - R_l)}}{k^2}$$

In this formula, $\boldsymbol{r}_j$, $\boldsymbol{p}_j$, and m represent the coordinates, momentum, and electronic mass, respectively. $\boldsymbol{R}_1$, $\boldsymbol{\pi}_1$, and M represent the analogous values for ions. N_1 and N_2 are the number of electrons and ions in a unit volume.

The wave function of the system is dependent on the variables $\boldsymbol{r}$, σ, $\boldsymbol{R}$, Σ (coordinates and spin of the electrons and ions), and on the supplementary variables Q_k ($k \leqslant k_0$) which describe the collective motion of the particles. For the time being, we won't define k_0 except to say that it must be finite. We impose the following subsidiary condition on the wave function:

$$Q_k \, \varphi(\boldsymbol{r}_1, \sigma_1, \ldots \boldsymbol{R}_{N_2}, \Sigma_{N_2}, t/\ldots Q_k \ldots) = 0 \qquad (2)$$

The subsidiary condition (2) and operator (1) have been operated on by the unitary transformation

$$U = e^{-\frac{i}{h}\sum\limits_{k,\,k\leqslant k_0} P_k \varrho_k} \tag{3}$$

where

$$P_k = -ih\frac{\partial}{\partial Q_k}$$

and

$$\varrho_k = -e\sum_{j/1}^{N_1} e^{-ikr_j} + Ze\sum_{l/1}^{N_2} e^{-ikR_l} \tag{4}$$

is the Fourier component of the charge density operator,

$$\varrho(r) = -e\sum_{j/1}^{N_1}\delta(r - r_j) + Ze\sum_{l/1}^{N_2}\delta(r - R_l). \tag{5}$$

The transformed subsidiary condition has the form

$$(Q_k - \varrho_k)\,\varphi' = 0 \text{ for } k\leqslant k_0;\quad Q_k\,\varphi' = 0 \text{ for } k > k_0 \tag{6}$$

where

$$\varphi' = U\varphi.$$

With the help of this condition, the Hamiltonian can be written in the form

$$H' = UHU^{+} = \sum_{j/1}^{N_1}\frac{p_j^2}{2m} + \sum_{l/1}^{N_2}\frac{\pi_l^2}{2M} + \frac{1}{2}\sum_{j_1\neq j_2}\sum_{k,\,k>k_0}\frac{4\pi e^2}{k^2}e^{ik(r_{j_1} - r_{j_2})} +$$

$$+\frac{1}{2}\sum_{l_1\neq l_2}\sum_{k,\,k>k_0}\frac{4\pi Z^2e^2}{k^2}e^{ik(R_{l_1} - R_{l_2})} - \sum_{j,\,l}\sum_{k,\,k>k_0}\frac{4\pi Ze^2}{k^2}e^{ik(r_j - R_l)} +$$

$$+\left(\frac{e^2N_1}{2m} + \frac{Z^2e^2N_2}{2M}\right)\sum_{k,\,k\leqslant k_0}k^2\,P_k\,P_{-k} + 2\pi\sum_{k,\,k\leqslant k_0}\frac{1}{k^2}\,Q_k\,Q_{-k} - \tag{7}$$

$$-2\pi\,e^2\,N_1\sum_{k\leqslant k_0}\frac{1}{k^2} - 2\pi\,Z^2\,e^2\,N_2\sum_{k\leqslant k_0}\frac{1}{k^2} -$$

$$-\frac{ie}{m}\sum_{j,\,k,\,k\leqslant k_0}e^{-ikr_j}\,P_{k\,k}\left(\frac{hk}{2} - p_j\right) + \frac{iZe}{M}\sum_{l,\,k,\,k\leqslant k_0}e^{-ikR_l}\,P_{k\,k}\left(\frac{hk}{2} - \pi_l\right) -$$

$$-\frac{e^2}{2m}\sum_{\substack{j,\,k_1,\,k_1\leqslant k_0\\ k_2,\,k_2\leqslant k_0\\ k_1+k_2\neq 0}}P_{k_1}P_{k_2}(k_1k_2)\,e^{-i(k_1+k_2)r_j} - \frac{Z^2e^2}{2M}\sum_{\substack{l,\,k_1,\,k_1\leqslant k_0\\ k_2,\,k_2\leqslant k_0\\ k_1+k_2\neq 0}}P_{k_1}P_{k_2}(k_1k_2)\,e^{-i(k_1+k_2)R_l}$$

Since in transformation (3), the summation is carried out over $k \leqslant k_0$, we obtain the following terms in the Hamiltonian (7):

a) terms dependent on the coordinates and momenta of the particles; these refer to the particles whose mass and spin are the same as the particles considered at the outset, but whose charge is such that the potential energy of their interaction is the short-range Coulomb interaction energy.

b) terms dependent only on variables Q_k, P_k, $(k \leqslant k_0)$; these give the energy for collective oscillations which follow Bose-Einstein statistics;

c) terms dependent on P_k $(k \leqslant k_0)$ and on the coordinates of the particles; these represent the energy connected with the interaction of the particles with the collective oscillations.

3. Generalization of the Collective Description Method

We now pass to the generalization of the collective description method for a system of electrons and ions.

We begin with the Hamiltonian,

$$\mathcal{H} = \sum_{j|1}^{N_1} \frac{\left(\boldsymbol{p}_j + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}_j)\right)^2}{2m} + \sum_{l|1}^{N_2} \frac{\left(\boldsymbol{\pi}_l - \frac{Ze}{c}\boldsymbol{A}(\boldsymbol{R}_l)\right)^2}{2M} + \frac{1}{8\pi}\int E^2(\boldsymbol{r})\, dv -$$

$$- 2\pi N_1 e^2 \sum_k \frac{1}{k^2} - 2\pi N_2 Z^2 e^2 \sum_k \frac{1}{k^2} \tag{8}$$

$\boldsymbol{E}(\boldsymbol{r})$ is the intensity of the longitudinal electric field, while $\boldsymbol{A}(\boldsymbol{r})$ is the vector potential of this field.

We now set up the Fourier representation

$$\boldsymbol{A}(\boldsymbol{r}) = (4\pi c^2)^{\frac{1}{2}} \sum_k q_k \boldsymbol{\varepsilon}_k\, e^{i\boldsymbol{k}\boldsymbol{r}} \tag{9}$$

$$\boldsymbol{E}(\boldsymbol{r}) = -(4\pi)^{\frac{1}{2}} \sum_k \dot{q}_k \boldsymbol{\varepsilon}_k\, e^{i\boldsymbol{k}\boldsymbol{r}} = (4\pi)^{\frac{1}{2}} \sum_k p_{-k} \boldsymbol{\varepsilon}_k\, e^{i\boldsymbol{k}\boldsymbol{r}}$$

$\boldsymbol{\varepsilon}_k$ is a unit vector having the direction of vector $\boldsymbol{k}$.

To ensure that $\boldsymbol{A}(\boldsymbol{r})$ and $\boldsymbol{E}(\boldsymbol{r})$ are real, we take

$$q_k = -q^+_{-k}\,, \quad p_k = -p^+_{-k} \tag{10}$$

The system wave function Φ must satisfy the subsidiary condition,

$$(\operatorname{div} \boldsymbol{E}(\boldsymbol{r}) - 4\pi\varrho(\boldsymbol{r}))\,\Phi = 0 \tag{11}$$

where $\varrho(\boldsymbol{r})$ is expressed by (5).

The above condtion can be written for the respective Fourier components

$$\left\{\sum_k \left[(4\pi)^{\frac{1}{2}} ikp_{-k} - 4\pi\varrho_k\right] e^{ikr}\right\} \Phi = 0 \tag{12}$$

or

$$\left(p_{-k} + i\left(\frac{4\pi}{k_2}\right)^{\frac{1}{2}} \varrho_k\right) \Phi = \Omega_k \Phi = 0 \tag{13}$$

where ϱ_k is given by equation (4).

Inserting expansion (9) into (8), we obtain

$$\mathcal{H} = \sum_{j/1}^{N_1} \frac{p_j^2}{2m} + \sum_{l/1}^{N_2} \frac{\pi_l^2}{2M} - \frac{1}{2}\sum_k p_k p_{-k} - \left(\frac{2\pi e^2 N_1}{2m} + \frac{2\pi Z^2 e^2 N_2}{2M}\right) \sum_k q_k q_{-k} +$$

$$- 2\pi N_1 e^2 \sum_k \frac{1}{k^2} - 2\pi N_2 \sum_k \frac{1}{k^2} +$$

$$+ \frac{e}{m}(4\pi)^{1/2} \sum_{k,j} \varepsilon_k \left(p_j - \frac{hk}{2}\right) q_k e^{ikr_j} - \frac{Ze}{M}(4\pi)^{1/2} \sum_{k,l} \varepsilon_k \left(\pi_l - \frac{hk}{2}\right) q_k e^{ikR_l} + \tag{14}$$

$$+ \frac{2\pi e^2}{m} \sum_{\substack{k_1, k_2, j \\ k_1 + k_2 \neq 0}} \varepsilon_{k_1} \varepsilon_{k_2} q_{k_1} q_{k_2} e^{i(k_1 + k_2) r_j} + \frac{2\pi Z^2 e^2}{M} \sum_{\substack{k_1, k_2, l \\ k_1 + k_2 \neq 0}} \varepsilon_{k_1} \varepsilon_{k_2} q_{k_1} q_{k_2} e^{i(k_1 + k_2) R_l}$$

The subsidiary condition (13) and the Hamiltonian (14) can undergo the unitary transformation,

$$S = e^{-\frac{i}{h} \sum_{k,\, k > k_c} \left(\frac{4\pi}{k^2}\right)^{1/2} q_k \varrho_{-k}} \tag{15}$$

In this transformation, the summation is taken over k for $k > k_c$. The value k_c (see BP 1952, 1953) is the quantum mechanical analogue to the value $k_D \simeq \frac{1}{\lambda_D}$ appearing in the classical plasma theory, where λ_D is the Debye length.

We will use the following transformations:

$$p_j \to S p_j S^+ = p_j - e(4\pi)^{1/2} \sum_{k,\, k > k_c} q_k \varepsilon_k e^{ikr_j}$$

$$\pi_l \to S\pi_l S^+ = \pi_l + Ze(4\pi)^{1/2} \sum_{k,\, k > k_c} q_k \varepsilon_k e^{ikR_l}$$

$$p_k \to S p_k S^+ = p_{-k} - i\left(\frac{4\pi}{k^2}\right)^{1/2} \varrho_{-k} \quad \text{for} \quad k > k_c \tag{16}$$

$$S p_{-k} S^+ = p_{-k} - i\left(\frac{4\pi}{k^2}\right)^{1/2} \varrho_k \quad \text{for} \quad k > k_c \tag{17}$$

The subsidiary condition (13) can be written in the form

$$S\Omega_k S^+ S\Phi = 0 \tag{18}$$

If we now set $S\Phi = \Phi'$, then after taking note of (17), the subsidiary condition will have the form

$$p_{-k}\,\Phi' = 0 \quad \text{for } k > k_c \tag{19}$$

and

$$\left(p_{-k} + i\left(\frac{4\pi}{k^2}\right)^{1/2} \varrho_k\right) \Phi' = 0 \quad \text{for } k \leqslant k_c$$

We now apply transformation (15) to the Hamiltonian (14). Using (16) and (19), we get

$$\begin{aligned}
\mathcal{H}' = S\mathcal{H}S^{+} = &\sum_{j/1}^{N_1} \frac{p_j^2}{2m} + \sum_{l/h}^{N_2} \frac{\pi_l^2}{2M} + \frac{1}{2} \sum_{j_1 \neq j_2} \sum_{k,\, k>k_c} \frac{4\pi e^2}{k^2}\, e^{ik(r_{j_1} - r_{j_2})} + \\
&+ \frac{1}{2} \sum_{l_1 \neq l_2} \sum_{k,\, k>k_c} \frac{4\pi Z^2 e^2}{k^2}\, e^{ik(R_{l_1} - R_{l_2})} - \sum_{j,l} \sum_{k,\, k>k_c} \frac{4\pi Z e^2}{k^2}\, e^{ik(r_j - R_l)} - \\
&- \frac{1}{2} \sum_{k,\, k<k_c} p_k\, p_{-k} - \left(\frac{2\pi e^2 N_1}{m} + \frac{2\pi Z^2 e^2 N_2}{M}\right) \sum_{k,\, k\leqslant k_c} q_k\, q_{-k} - \\
&- 2\pi N_1 e^2 \sum_{k\leqslant k_c} \frac{1}{k^2} - 2\pi N_2 Z^2 e^2 \sum_{k\leqslant k_c} \frac{1}{k^2} + \\
&+ \frac{e}{m}(4\pi)^{1/2} \sum_{k,\, k<k_c} \varepsilon_k \left(p_j - \frac{h_k}{2}\right) q_k\, e^{ikr_j} - \frac{Ze}{M}(4\pi)^{1/2} \sum_{k,\, k<k_c} \varepsilon_k \left(\pi_l - \frac{h_k}{2}\right) q_k\, e^{ikR_l} + \\
&+ \frac{2\pi e^2}{m} \sum_{\substack{j,\, k_1,\, k_1 \leqslant k_c \\ k_2,\, k_2 \leqslant k_c \\ k_1 + k_2 \neq 0}} \varepsilon_{k_1} \varepsilon_{k_2} q_{k_1} q_{k_2}\, e^{i(k_1 + k_2) r_j} + \frac{2\pi Z^2 e^2}{M} \sum_{\substack{l,\, k_1,\, k_1 \leqslant k_c \\ k_2,\, k_2 \leqslant k_c \\ k_1 + k_2 \neq 0}} \varepsilon_{k_1} \varepsilon_{k_2} q_{k_1} q_{k_2}\, e^{i(k_1 + k_2) R_l}\,.
\end{aligned} \tag{20}$$

Since in transformation (15) the summation was taken over $k > k_c$, components q_k and p_k will appear in the Hamiltonian for $k = |k| \leqslant k_c$. In the paper (BP 1953) they were called collective coordinates and describe the collective field. Condition (19) says that the source of this field is part of the charge, since the component ϱ_k appears in the condition only for $k \leqslant k_0$ $\left(\varrho(r) = \sum_{k,\, k\leqslant k_c} \varrho_k\, e^{ikr} + \sum_{k,\, k>k_c} \varrho_k\, e^{ikr}\right)$.

The remaining part of the charge relates to those particles which have the same mass and spin as the particles first considered, but whose charge is such that the particles interact with each other through the intermediary of a screened Coulomb potential.

The terms appearing in the Hamiltonian can be arranged in the following manner:

A) terms dependent only on the coordinates and momenta of the particles (with their charge and interaction as above);

B) terms dependent only on variables q_k and p_k $(k \leqslant k_c)$; these represent the energy of the collective oscillations which follow Bose-Einstein statistics;

C) terms dependent on q_k $(k \leqslant k_c)$ and the coordinates of the particles; these represent the energy due to the interaction of the particles with the collective oscillations.

4. *Equivalence of Both Methods*

We will now show that by a change of variables, the Hamiltonian (7) can be transformed into the Hamiltonian (20) and subsidiary condition (6), into subsidiary condition (19).

We write condition (19) in the form

$$\left(i\left(\frac{k^2}{4\pi}\right)^{1/2} p_{-k} - \varrho_k\right) \Phi' = 0 \tag{19'}$$

In addition, using the relation

$$e^{-ikr} f(\boldsymbol{p}) = f(\boldsymbol{p} + h\boldsymbol{k})\, e^{-ikr} \tag{21}$$

we write the Hamiltonian in the form

$$\begin{aligned} H' = {} & \sum_{j/1}^{N_1} \frac{p_j^2}{2m} + \sum_{l/1}^{N_2} \frac{\pi_l^2}{2M} + \frac{1}{2} \sum_{j_1 \neq j_2} \sum_{k,\, k>k_0} \frac{4\pi e^2}{k^2}\, e^{ik(r_{j_1} - r_{j_2})} + \\ & + \frac{1}{2} \sum_{l_1 \neq l_2} \sum_{k,\, k>k_0} \frac{4\pi Z^2 e^2}{k^2}\, e^{ik(R_{l_1} - R_{l_2})} - \sum_{j,l} \sum_{k,\, k>k_0} \frac{4\pi Ze^2}{k^2}\, e^{ik(r_j - R_l)} + \\ & + 2\pi \sum_{k,\, k\leqslant k_0} \frac{1}{k^2}\, Q_k\, Q_{-k} + \left(\frac{e^2 N_1}{2m} + \frac{Z^2 e^2 N_2}{2M}\right) \sum_{k,\, k\leqslant k_0} k^2 P_k\, P_{-k} + \\ & - 2\pi N_1 e^2 \sum_{k \leqslant k_0} \frac{1}{k^2} - 2\pi N_2 Z^2 e^2 \sum_{k \leqslant k_0} \frac{1}{k^2} + \\ & - \frac{ie}{m} \sum_{j,\, k,\, k\leqslant k_0} P_{-k}\, \boldsymbol{k} \left(\boldsymbol{p}_j - \frac{h\boldsymbol{k}}{2}\right) e^{ikr_j} + \frac{iZe}{M} \sum_{l,\, k,\, k\leqslant k_0} P_{-k}\, \boldsymbol{k} \left(\boldsymbol{\pi}_l - \frac{h\boldsymbol{k}}{2}\right) e^{ikR_l} + \\ & - \frac{e^2}{2m} \sum_{\substack{j,\, k_1,\, k_1 \leqslant k_0 \\ k_2,\, k_2 \leqslant k_0 \\ k_1 + k_2 \neq 0}} P_{-k_1} P_{-k_2} (\boldsymbol{k}_1\, \boldsymbol{k}_2)\, e^{i(k_1 + k_2) r_j} - \frac{Z^2 e^2}{2M} \sum_{\substack{l,\, k_1,\, k_1 \leqslant k_0 \\ k_2,\, k_2 \leqslant k_0 \\ k_1 + k_2 \neq 0}} P_{-k_1} P_{-k_2} (\boldsymbol{k}_1\, \boldsymbol{k}_2)\, e^{i(k_1 + k_2) R_l} \end{aligned} \tag{22}$$

If in Hamiltonian (22) we change to the new variables q_k, p_k

$$Q_k \to i\left(\frac{k^2}{4\pi}\right)^{1/2} p_{-k}\,; \quad P_k \to i\left(\frac{4\pi}{k^2}\right)^{1/2} q_{-k} \tag{23}$$

and let

$$k_0 = k_c \tag{23'}$$

then this Hamiltonian will be identical with Hamiltonian (20). Similarly, subsidiary condition (6) will be identical with (19').

Transformation (23) conserves the commutation relation

$$Q_k P_k - P_k Q_k \to -(p_{-k} q_{-k} - q_{-k} p_{-k}) = ih \tag{24}$$

Thus, there must exist a unitary transformation $\tilde{S}$ which transforms

$$(Q_k, P_k) \to (q_k, p_k)$$

This means

$$\tilde{S} Q_k \tilde{S}^+ = i \left(\frac{k^2}{4\pi}\right)^{1/2} p_{-k} \; ; \; \tilde{S} P_k \tilde{S}^+ = i \left(\frac{4\pi}{k^2}\right)^{1/2} q_{-k} \tag{25}$$

As can be shown, this transformation has the form

$$\tilde{S} = exp \left\{ -\frac{i}{2h} \sum_k \left[\left(Q_k - i \left(\frac{k^2}{4\pi}\right)^{1/2} p_{-k} \right) \left(i \left(\frac{4\pi}{k^2}\right)^{1/2} q_{-k} - P_k \right) + \right.\right.$$

$$\left.\left. + \left(i \left(\frac{4\pi}{k^2}\right)^{1/2} q_{-k} - P_k \right) \left(Q_k - i \left(\frac{k^2}{4\pi}\right)^{1/2} \right) \right] + \text{const} \right\} . \tag{26}$$

Therefore, if we apply transformation (26) to the Hamiltonian (22) and subsidiary condition (6) (supplementary variables method), we obtain the Hamiltonian (20) and subsidiary condition (19') (collective description method).

Applying transformation (26) to transformation (3) and taking into account (23'), we obtain

$$\tilde{U} = \tilde{S} U \tilde{S}^+ = e^{\frac{1}{h} \sum_{k,\, k \leqslant k_c} \left(\frac{4\pi}{k^2}\right)^{1/2} q_{-k} \varrho_k} = e^{\frac{1}{h} \sum_{k,\, k \leqslant k_c} \left(\frac{4\pi}{k^2}\right)^{1/2} q_k \varrho_{-k}} \tag{3'}$$

Comparing (3') with transformation (15), we see that if the summation is taken over all k then transformation (3') would be the inverse of transformation (15). Hamiltonian (1) representing the energy of the particles interacting with each other by means of a Coulomb potential, would after application of transformation (3') change into Hamiltonian (14) or (8), describing the energy of the particles moving in the longitudinal electric field and the energy of this field. On the other hand, application of transformation (15) to either Hamiltonian (8) or (14) would lead to the form (1).

On the basis of experimental investigation of plasma (see, for example, Langmuir and Tonks 1929), it was postulated that the Hamiltonian should contain variables (Q_k, P_k) for $k \leqslant k_0$, or (q_k, p_k) for $k \leqslant k_c$. This corresponds, in the first method, to taking the summation in transformation (3) over $k \leqslant k_0$, and in the second method, to taking the summation in transformation (15) over $k > k_c$. As a result, the complete transition from (1) to (14) in the first case and from (14) to (1) in the second case,

doesn't take place. It could be said that by taking $k_0 = k_c$, both methods, which start from different, but equivalent Hamiltonians, meet at the half-way point.

The equivalence which has been established for both methods permits a physical interpretation of variables Q_k; they are the Fourier components of the divergence of vector **E**. In addition, we can now, on the basis of the examination given by BP (1952, 1953) and Pines (1953) estimate the value of k_0 in Zubarev's theory. Finally, we note that if the summation in transformation (3) was taken over all k, the method of supplementary variables would be the inverse to the method of elimination of the longitudinal electric field from the Hamiltonian.

I would like to thank Professor M. Günther, Professor R. S. Ingarden, and Dr. A. Rybarski for helpful discussions.

КРАТКОЕ СОДЕРЖАНИЕ

С. Галясевич. *Об эквивалентности метода Зубарёва и метода Бохма и Пинеса для систем, состоящих из двух видов частиц.*

В этой работе обобщён метод коллективной описи, представленной Бохмом и Пинесом (Phys. Rev., **92**,609 (1953), для систем, состоящих из двух видов частиц (электронов и ионов). Результат сравнен с результатами автора (Галясевич, Acta phys. Pol.) содержащими обобщение на этот же случай метода переменных добавочных Зубарёва (Z.E.T.F. **25**, 548 (1953)). Обнаружена эквивалентность обоих методов. Представлена унитарная трансформация, которая преобразует гамильтониан, полученный одним методом, в гамильтониан, полученный другим методом.

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A DISCUSSION ON BILOCALITY

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(Received November 28, 1955)

The conceptual situation in contemporary theoretical physics is discussed. The position and displacement operators x_μ, d_μ form an abstract algebra constituting a natural generalization of the concept of space-time (space-time quantization), and a starting point for a bilocal theory. The canonical transformations of the operators x_μ, d_μ have to play an important role. The linear subgroup of the group of canonical transformations is unimodular. This offers a simple and natural interpretation of isospin. The problem of rest-mass is also discussed. Mass consists of two parts: a kinetic (mechanical) mass and a field mass. The former is connected with the internal motion of the bipoint particle. For higher states of internal motion the self-mass is probably negligeable in comparison with the kinetic mass.

Introduction

The first note on non-local (bilocal) field theory was published by Yukawa in form of a letter to the Editor of Physical Review (1950). Soon afterwards an enthusiastic review by Born appeared in Nature, wherefrom we cite:

„The purpose of the following note is to direct the attention of physicists to a short paper by H. Yukawa which has recently appeared. It seems to me to contain a most important advance in the theory of elementary particles, based on an extremly simple idea... This idea is so elementary and simple that I expect important results from further development."

Although six years elapsed since that time, and a few further papers on bilocal theory have been published by some authors, decisive confirmations of the correctness of the ideas and implications of this theory are still lacking. The general interest in this new theory seems to be weak. It seems that a great majority of physicists regards the bilocal theory to be nothing else but one more attempt, in the flood of attempts of modifying the contemporary not very satisfactory field theory, appearing every year in the literature. These attempts tend in various directions and obviously a great majority of them will suffer the usual fate: oblivion. Regarding the Yukawa theory to be not distinguished by anything particular in comparison with dozens of competing proposals, the physicists would be a little surprized if some day it turned out that

it is just this theory finding confirmation by experimental results in the domain of elementary particles. In my opinion one should be rather astonished in the reverse case: If the future experimental results would not confirm the implications of bilocality.

Believing that the ideas of Born and Yukawa are fruitful I decided to write this article, in a form rather unusual for a scientific paper. This form (reminding rather the forms used by novelists) seems to suit best my purpose: to show that (in the spirit of the past developments of theoretical physics) the bilocal theory may be regarded as a very natural and almost unavoidable generalization of the contemporary theory, and that its future success is equally plausible as was e. g. the case with the theory of relativity, in the days of its birth, when its confirmation by experiment was not yet decisive.

It is a fundamental principle of empirical science that experiment is its only criterion. It is admitted generally that, at any stage of human knowledge, the theoretical possibilities are not unique. Several logically consistent explanations of facts seem to correspond equally well to reality until the discovery of new facts enables one to decide which of the possibilities have to be excluded. History of science shows that people are not aware of all theoretical possibilities consistent with the known facts but usually see only a single possibility, often the wrong one. Only in view of quite a new fact, sharply contradicting their theoretical views, they start to look for new conceptions and, sooner or later, discover them. In this way the theory of relativity and the quantum mechanics have been discovered.

We have no intention to deny the above views upon the relation of the theory to experiment but wish to call attention to the circumstance that sometimes the experimental evidence forming the basis for a theory was rather scarce, whereas the theory itself constituted a powerful weapon for predicting, qualitatively and quantitatively, a lot of new, yet unknown facts. We witnessed several times this astonishing situation that theory outdistanced experimental facts.

Some theoretical conceptions seem so obvious and clear (when already formulated by someone) that, if they came to ones mind before the discovery of the crucial fact (which induced and enabled their formulation), they could be propagated with a great amount of confidence in their correctness. This concerns, above all, the special theory of relativity discovered by Einstein to account of the negative result of Michelson's experiment. One could well imagine that Einstein formulated his theory before the experiment of Michelson. Less probable, though not quite impossible would be the invention of quantum mechanics before the experimental discovery of the facts of a typically quantum character. The discovery of Planck was, however, to a certain degree, just of this type.

In view of such a situation I sometimes wonder what minimum of facts would be sufficient for the human intellect to reconstruct all most fundamental laws of contemporary theoretical physics? The results of my speculations were surprising even for myself. It seems that this minimum could be very scarce. In my opinion the knowledge of facts which already Newton knew and, additionally, a fact denied by

Newton that light consists of waves (satisfying the principle of superposition), would be sufficient.

We may try to imagine possible lines of reasoning of a fictitious super-theorist leading him, in a most straightforward way, to the invention of contemporary theoretical physics on the basis of the above mentioned minimum of facts. It seems to me that such a play of imagination is not a mere entertainment but may be of some value for the methodology of theoretical physics. It may help to detect those features which favour principally the already well established theoretical conceptions, independently of their numerous experimental confirmations. A better understanding of these features may promote future attempts at theoretical generalizations.

Two fictitious scientists reconstruct the modern theoretical physics

In order to make the whole affair psychologically more plausible let us assume that not a single but two theorists undertake this enormous task. Imagine they are living isolated from the rest of the civilized world. Assume they are endowed in the abilities of all mathematicians that ever lived on earth so that they are able to reconstruct easily any chapter of mathematics they need. But they are interested chiefly in theoretical physics. Unfortunately, they are not aware of the modern developments of experimental and theoretical physics, and in their library they have only Newton's Works and a treatise on optics, wherefrom they infer that light is a wave motion satisfying the principle of superposition. To fix attention assume they are brothers. The elder is called ***R***(elativist), the younger ***Q***(uantist).

Regarding Newton's mechanics for experimentally well established, ***R*** sets to work and invents the equations of Lagrange. He becomes convinced that covariance with respect to general transformations of coordinates is of great importance for theoretical physics. ***Q*** studies this work of his elder brother and undertakes the work of Hamilton, i. e. invents the variational principle of Hamilton and the canonical formalism. After having completed this work he becomes convinced that canonical transformations should play an important role in the future developments of theoretical physics. His mechanics is, in principle, a mechanics of systems of material points. He is aware of the atom hypothesis (known since antiquity) and assumes that his mechanics applies to those smallest indivisible particles. In view of a complete lack of information about the structure of atoms, (even electrostatic forces are not known to him) he identifies his atoms with what we call „elementary particles".

Meanwhile ***R*** turns to optics and, on the basis of the available optical treatise, writes down the wave equation

$$\Box \varphi = 0. \tag{1}$$

On this occasion he is struck, for the first time, by the analogy of the role played by the variable $x_0 = ct$ as compared with the variables x_1, x_2, x_3. Returning to the equations of Newton he begins to think about the principle of relativity in Newtons mechanics. Noticing that the wave equation (1) is not invariant with respect to Galilei trans-

formations he puts the problem as to the transformation under which (1) is invariant, and finds easily that the simplest group of transformations satisfying this requirement is the group of linear transformations preserving the quadratic form

$$x_1^2 + x_2^2 + x_3^2 - x_0^2 = \text{inv.} \tag{2}$$

This leads him directly to the discovery of the theory of relativity (at first the special one, and then the general). His method is quite similar to that of Einstein. He has no reasons to distrust his own method and believes that experiment will confirm the predictions of both theories of relativity. At the same time he became aware of the important role played by the group of general transformations of space-time coordinates.

As yet the „aprioristic" methods of both brothers seem to have no improbable features. It will be more difficult to convince the reader that by means of pure speculations Q will be able to discover quantum mechanics. However, assuming that ***Q*** is in fact exceptionally able (moreover he is the inventor of canonical formalism!), we will find it not incredible that, when contemplating the properties of classical Poisson brackets, he should notice that the same properties as the Poisson brackets have also the commutators. Thus, applying the well known arguments of Dirac he assumes the canonical commutation relations

$$[q_j, p_k] = ih\delta_{jk}. \tag{3}$$

The numerical value of the constant h is, of course, unknown to him. Having the commutation relations (3), it is not difficult for him to develop the rest of the quantum formalism and to propose a probabilistic interpretation.

Being not aware of any experimental facts in favour of a quantic character of physical events, he cannot be sure whether his quantum mechanics will correspond to reality better than classical mechanics. Both are covariant with respect to the group of canonical transformations, both constitute formalisms of admirable beauty and consistency. We shall not conceal a sad fact of a family quarrel: the elder brother ***R*** was not at all impressed by the achievement of his younger brother (who so much admired him) and called his attention to the unpleasant fact that quantum mechanics is difficult to be reconciliated with the ideas of relativity. Moreover, he stressed the fact that there are no a priori reasons in favour of quantum mechanics and against classical mechanics. ***Q*** liked the idea of quantization. He admitted that quantum mechanics is only one alternative (the other is classical mechanics) but this alternative is so interresting that it cannot be rejected on the only ground that it is more difficult. Fortunately the quarrel was not very serious, and ***Q*** succeeded in convincing his brother that it is worth while to examine quantum theory further and to try to reconciliate it with relativity. Since at that time ***R*** did not see any new interesting topics in the theory of relativity, he joined ***Q*** and both started a new branch of theoretical physics: quantum field theory. Led by a close analogy with quantum mechanics of systems with a finite number of degrees of freedom, they perform the quantization of the wave equation (1) (this may be interpreted as a system with infinitely many degrees of freedom).

They find relativistically covariant commutation relations for the field quantity taken at two arbitrary space-time points. Moreover, they find that the quantized field is equivalent to a set of particles (with vanishing rest mass). It was not difficult for them to generalize (1) to the Schrödinger-Gordon equation, corresponding to particles with a non-vanishing rest mass. They generalized their theory for the case where the field quantity is not scalar but tensor of any rank. Since we have no reason to assume that they are less able than Cartan, we believe that they find also spinors being a natural generalization of tensors. Thus, besides the Schrödinger-Gordon equation they investigate also the Dirac equation. Finding in this last case the energy to be not definite, they see a remedy in a new type of quantization using anticommutators instead of commutators. Finally, they prove that particles with integer (half-integer) spins are obliged to satisfy statistics which we used to call Bose-Einstein (Fermi-Dirac) statistics respectively.

In this way the main chapters of contemporary theoretical physics have been completed. Their theory concerns only the case of free fields. They did not succeed in formulating a theory of interaction except for a classical theory of gravitation. But we must not blame them fot it: Also our theory of interactions is far from being satisfactory in spite of the circumstance that our knowledge of experimental facts exceeds by far that of the two fictitious scientists.

It is only a fairy-tale, but it involves no irrational elements. The story about the two brothers contains nothing impossible. We were not obliged to assume them to possess any miraculous intellectual qualities, but assumed only that simple but right ideas used to come to their minds at right moments. This is not principally impossible although it occurs very seldom, as is known from the history of physics. At any rate, the arguments used by our two brothers have been selected exclusively out from the arguments used sometime by real scientists.

Two fictitious scientists criticize the present status of theoretical physics in the light of canonical transformations

R. I think, an utmost importance should be ascribed to the principle of equivalence stating that the laws of physics should be covariant under general transformations of space-time coordinates.

Q. I do not deny the importance of the group of these transformations, but I should like to add that, in my opinion, also the group of canonical transformations is to play a very important role. We know very well how essential this group is in classical mechanics. It is no less important in quantum mechanics. I don't know yet any experimental confirmations of quantum mechanics but I think the lack of evidence for the existence of quantic effects may be caused simply by the smallness of the constant h appearing in the formalism.

R. I confess frankly that I don't believe in the success of your quantum mechanics. The chance that quantum mechanics will agree with experiment is by far smaller than the chance that my theory of relativity will do so. The arguments in favour of the theory.

of relativity are by far more convincing than those in favour of quantum mechanics. I shall not repeat them now, I wish only to call your attention to the fact that the transition from non-relativistic to relativistic theory means a generalization of the group of transformations constituting a very natural and essential enrichment of physical contents, whereas by your transition from classical to quantum mechanics the group of canonical transformations is not generalized. Therefore your quantum mechanics is only an alternative besides classical mechanics whereas my relativistic theory is a generalization of the pre-relativistic theory.

Q. This point may be argued. Quantum mechanics may be regarded as a generalization of classical mechanics of a similar character as that from pre-relartivistic to relativistic theory. Both generalizations consist in an incorporation of a (dimensional) constant h or c into the theory. Similarly as in the limit $c \to \infty$ the relativistic theory goes over into the non-relativistic, also the quantum theory goes over, in the limit $h \to o$, in the classical theory. By the way, it has been always my favourite idea that in the future theory a third (dimensional) constant, e.g. a fundamental length l should be incorporated so that, in natural units $c = h = l = 1$, every physical quantity may be regarded as dimensionless.

R. This last idea seems very interesting. But, as we do not know yet how to incorporate it, let us return to our first topic. I should like to stress once more that I maintain my thesis that relativity possesses more features qualifying it to be a priori correct than quantum theory.

Q. I agree with you although I don't know whether our agreement follows from the same arguments. I would like to know better your point of view. Explain me, please, why do you think so, and then I shall explain why my opinion about the foundations of both theories leads to a similar conclusion.

R. I think that relativity and its implications are a priori more plausible since the transformations considered by the relativist concern, in the first instance, the points of space-time itself, and only in the second instance the models of physical objects (particles, fields) situated in space-time. On the other hand, the canonical transformations (which you regard to be so important) concern only the latter. Therefore the theory of relativity led us to a revision of our notions in a very fundamental question: about the structure of space-time, whereas quantum mechanics does not touch this structure at all. Thus, it may be stated that there exists a deep difference in the very subject of investigation of the two theories. Since relativity is concerned with the points of space-time itself, or, it regards space-time as a physical object, it is more elementary, and enables one to formulate most general laws, free of any specific features and hypothesis about the nature of physical objects (or events) which may be situated (or take place) at the points of space-time.

Q. Yet, canonical transformations are more general than transformations of coordinates only.

R. Particles and fields possess momenta whereas the points of space time itself possess no momenta. Since the properties of points of space time are more abstract

than the properties of any particles situated in space-time, my transformations have a more fundamental and universal character than yours. I don't think that you are right in ascribing such a fundamental significance to the group of canonical transformations.

Q. I promised to tel you why I regard quantum mechanics to be less strongly legitimated than relativity is. First of all it is founded not so firmly since it is only an alternative. The other alternative is classical mechanics and your arguments apply as well to quantum as to classical mechanics. We should not forget that the invention of quantum mechanics was strongly supported by classical mechanics. A criterion of validity of any one of the two alternatives (quantum or classical) is the validity of Newtonian mechanics for macroscopic events. We should not forget that, although we have not supported our deductions by experiments, the starting point of our speculation was the experimentally well founded Newtonian mechanics. My conviction of importance of the canonical transformations has not an a priori character. Most probably the idea of canonical transformations would never occur to me if I had not at hand the (experimentally well founded) mechanics of Newton. If finally it will turn out that not classical but quantum mechanics is true, its legitimation will be still the (asymptotic) validity of Newton's mechanics. The source of my knowledge is not my mind but those experimental facts which constituted once the basis of Newton's speculations. The way from experimental facts known to Newton to the most subtle quantum effects (as e.g. the fact that particles may possess only integer or half-integer spins, or the exclusion principle for identical particles with half integer spins) seems to be very long, but still these effects are deductions from the Newtonian experimental facts.

The objects of investigation of (classical or quantum) mechanics are models of physical bodies or fields given in experiment, and we have to know something about them from experiment. It is impossible to invent all their properties a priori. Therefore my mechanics cannot be legitimated a priori with such amount of certainty as relativity which, as you told me, is concerned in the first instance not with mechanical models but with abstract points of space-time. This superiority of the theory of relativity is however only apparent since what would be the value of such a theory if finally we did not introduce into its framework (space-time) some mechanical models? It would be an empty, useless scheme.

R. From what you have told me one could easily draw a false conclusion that relativity is concerned with some abstractions having nothing to do with experiment. We are not allowed to say that, on the one hand, we have models of physical objects given in experiment, and on the other hand, we have the model of space beyond experiment. This would not be right. The theory of relativity regards the space-time also as a physical object and its abstract considerations concern the model (or models) of this physical object.

With the rest of your arguments about the origin of all your knowledge I agree. But I should like to add that also my knowledge is based on experiment and originates mostly in Newtons mechanics. I don't think that the idea of relativity would ever occur

to me if I did not know Newton's mechanics with its invariance with respect to Galilei transformation. Or, could I discover the Lorentz transformation without the knowledge of the wave equation (1) which is also based on experimental facts? In this respect the subjects of our theories are not different. The knowledge of both of us originated in experiment. When pointing out a difference in the fundamentality of both theories, I referred to a more subtle difference in the degree of elementarity and generality. I maintain my statement that your mechanics (classical or quantum) is lacking something which distinguishes my theory of relativity. Your mechanics seems to be too specific, too little abstract. Your models constitute a foreign element in the physical space-time, in contradistinction to a „par excellence" relativistic model provided by the gravitational field. This field is joined organically with space-time itself. I would feel happy only if you could present a new version of mechanics whose models were intimately connected with space-time itself. Only such mechanics would be elementary and fundamental enough to be qualified as almost a priori correct.

Q. Oh, I begin to understand your point of view. Our discussion was of valuable help for me since it enabled me to realize that mechanics will not reach the same degree of certainty as relativity until the objects of both theories be identical. In other words, similarly as in the theory of relativity, the main object of investigation of mechanics should be space-time itself. In particular, from the point of view of the relativist, quantum mechanics does not seem to be fundamental enough, since it does not concern space-time as such. Your arguments against quantum mechanics may be summarized as follows: you pointed out that a quantization of space-time itself is unknown. It seems to me that such a quantization may exist and I suspect that it has something to do with my favourite idea about the existence of a fundamental length l. Let me think it over.

Q. (after a while): The opinion that quantization concerns only models of physical objects in space-time but not space-time itself seems to be wrong. This opinion means too narrow an angle of view upon the true meaning of quantization. It is well known that the momentum p_j canonically conjugate to the coordinate q_j may be represented as

$$p_j = i\,h\,d_j, \tag{4}$$

where d_j is the operator of infinitesimal displacement of the coordinate q_j. The notion of infinitesimal displacement has the abstract character required, and, contrary to the notion of momentum, it is a purely mathematical concept (having nothing to do with mechanical concepts such as, e.g., the mass). The very fact that at the bottom of the notion of momentum lies another abstract concept, will be all important. Let us consider, for simplicity, a space-time with a vanishing curvature and introduce a pseudo-cartesian system of coordinates x_μ. A natural generalization of the notion of space-time will be an operator space where the x_μ are regarded as operators $\boldsymbol{x}_\mu$, whose counterparts are the infinitesimal displacement operators $\boldsymbol{d}_\mu$. Their operator character is determined by the commutation relations

$$[\boldsymbol{x}_\mu, \boldsymbol{d}_\nu] = \delta_{\mu\nu}, \quad [\boldsymbol{x}_\mu, \boldsymbol{x}_\nu] = [\boldsymbol{d}_\mu, \boldsymbol{d}_\nu] = 0. \tag{5}$$

If $\vec{x}$ is a real operator and x_4 an imaginary operator then $\vec{d}$ is imaginary and d_4 is real. The generalization of the classical space to the operator space (x_μ, d_ν) is nothing else but quantization of space-time.

R. Has this quantization anything to do with the constant h?

Q. Nothing at all, but it is all right. In the relativistic theory one used to equate the dimensions by putting $x_4 = ix_0 = ict$. Then all the x_μ have the dimension of length and the d_μ the dimension of reciprocal length. Similarly we should equate now the dimensions of the objects of quantized space-time, i.e. introduce a new constant l of dimension of length, and consider new quantities $l^2 d_\mu$ instead of d_μ. Thus, we have a very natural way of introducing a fundamental length into the theory. It seems plausible that in the future theory the three fundamental constants c, h, l will play an equally important role. The operators x_μ and $l^2 d$ are objects of an algebra with the operations of addition, multiplication of operators by complex numbers, and multiplication of operators by operators. This last type of multiplication is associative but not commutative. The replacement of classical coordinates by operators may be regarded as a transition to a higher level of abstraction. This procedure may be considered without any connection to the models of physical systems and to the constant h (having a dynamical meaning). The role of the constant h is taken over by the constant l having a geometrical meaning since the abstract commutation relations should be written in the form

$$[x_\mu\, l^2 d_\nu] = l^2 \delta_{\mu\nu},\ [x_\mu, x_\nu] = [l^2 d_\mu\, l^2 d_\nu\, ,] = 0. \qquad (5')$$

R. However, the question arises whether such a procedure has anything to do with what we used to call quantization?

Q. I think we should call it quantization, but, to distinguish it from the traditional h-quantization, introduce the name l-quantization. The operators x_μ, d_ν span a generalized Hilbert space. We may introduce a basis in this space and represent the operators by matrices. E.g. using the eigen-vectors $|\, x' >$ of the operators x_μ to their eigenvalues x'_μ, as a basis, any operator O may be represented as a matrix $< x' \mid O \mid x'' >$, i.e. as a function of a pair of points $O(x', x'')$. Thus, we have operators, eigenvalues, commutation relations, Hilbert space, representations, and transformation theory, i.e. all the formal elements of a quantum theory.

R. But what is the relation between this l-quantization and the traditional h-quantization?

Q. l-quantization is a quantization of space-time, and is primary as compared to the quantization of specific mechanical models. I used the word „primary" in a similar sense as we have used it when stating that relativisation of space-time (regarding the space-time coordinates x_μ as a four-vector, etc ...) is a primary fact in comparison with a relativisation of physical models. This last means that we ascribe a suitable transformation character to the physical quantities characterizing the model. On the other hand, the quantization of mechanical models (h-quantization) is also a secondary fact consisting in a suitable attachment of physical quantities (characterizing the model)

to the elements of quantized space-time. This attachment is performed with the aid of the rule (4).

R. In natural geometrical (kinematical) units $c = l = 1$ the constant h has a dimension of mass. It appears only at this stage when we want to establish a connection between the mechanical models and the quantized space-time. I am convinced by your arguments and see that I have to withdraw my objections against quantum theory. Quantization appears in a new light now, and seems to be equally fundamental a procedure as relativization. In view of the fact that physical space is an operator space, my former statement that „the particles possess momenta, whereas the points of space-time possess none" is no more right. Indeed, it is just the displacement operator $\boldsymbol{d}_\mu$ which may be regarded as a substitute of the notion of momentum canonically conjugate to $\boldsymbol{x}_\mu$. Yes. If it is so, it will be possible to speak as well about canonical transformations in connection with space-time as such. Now I accept also your view that canonical transformations are equally fundamental but more general than the transformations of the space-time points alone. The transformations of the (special and general) theory of relativity are only a special case of a more general group of canonical transformations. The operator space (or algebra) admits a more general group of transformations than the classical space and offers new possibilities of interesting generalizations. The traditional theory of relativity may be regarded as a theory on a Lagrangean level whereas now, as it seems, it may be generalized to a Hamiltonian level.

Q. Let us resume our discussion. In my opinion the most important result of our discussion is as follows: The theory of relativity as well as the quantum theory are both fundamental theories since the object of their investigations is, in the first instance, the structure of space-time itself. The theory of relativity states that the physical space is a four-dimensional curved manifold with a signature $+++-$ and, in the special case of vanishing curvature it becomes a pseudo-euclidean manifold wherein inertial systems of coordinates may be introduced. The transition from one inertial system of coordinates to another is given by the Lorentz transformation. The role of the time coordinate became similar to the roles of space coordinates upon equating their dimensions with the aid of a universal constant c. On the other hand, quantum theory generalizes the classical Minkowski space into an operator space (algebra) with the fundamental elements x_μ, d_μ. The role of d_μ becomes similar to the role of x_μ upon equating their dimensions with the aid of another universal constant l. In quantized space-time transformations of a more general type — the canonical transformations — are possible. In the second instance we may go over from the properties of space-time itself to the properties of models of physical objects. These objects are situated in space-time and have to have properties conformal with the structure of the space-time itself. Relativization of physical objects consists in ascribing to them a covariant meaning with respect to the transformations of the theory of relativity while their quantization consists in a „location" of these models in the quantized space-time, i.e. in an attachment of physical quantities characterizing the models (their coordinates, momenta, etc...) to the elements of quantized space-time. This coordination is achieved

by means of a simple rule (4) or its straightforward generalizations. The physical quantities have to have a covariant meaning with respect to transformations of elements of the quantum space-time, i.e. also with respect to canonical transformations.

It should be noticed that a direct transition to a higher level of abstraction (from the properties of models of physical objects to the properties of space-time itself) was possible only for quantum mechanics, but not for classical mechanics. We infer that only quantum mechanics possesses a fundamental justification similar to an „a priori justification" of relativity. We have no more two equally plausible possibilities (classical or quantum), but only a single one: quantum theory. Now, I believe that agreement with experiment will be reached by the more fundamental and more elementary quantum mechanics and not by the classical mechanics. The latter can be only an approximation valid for macroscopic objects.

R. A twofold success has been achieved. You have introduced into physical theory a fundamental length in such a natural way that I can't imagine any other, more elementary, and, at the same time you succeeded in establishing the superior character of quantum theory as compared with classical theory.

Our discussion has thrown also some light upon the meaning of the words such as „a priori sure", „a priori justified", used in our discussion rather intuitively. Of course, an absolutely „aprioristic" knowledge does not exist at all. Every knowledge is based on experimental facts. In particular, the knowledge of both of us is founded on the experimental facts already known to Newton. He derived from these facts some important consequences, and we have succeeded in deriving some more. I should like to know what distinguishes our work from mere guessing and what decides that our reasonings may be qualified as plausible (if not as certain). It seems to me that it is an ability of forming abstract notions and generalizations so characteristic for the human intellect. The methods of obtaining abstract notions useful in theoretical physics are exactly those used by Galileo and Newton. It seems to me that we have used consequently the same methodology. Having some facts at our disposal we tried to discern what is specific and what is universal in them. The notion of covariance with respect to some transformations played a particular role in our speculations. The point was no more and no less but to pick up these properties of physical objects which are independent of the mode of description. We ascribed a particular significance to the properties of space-time itself. This is natural since, in a certain sense, the physical space may be regarded as an essence of those properties which are common for all physical objects or events.

An attempt of generalizing the field theory

Q. Let us consider whether the method, which enabled us to derive the whole theoretical physics (such as is known to us at the present moment), may be applied further to give some more informations about the general laws of physics. I would like to know whether the idea of quantized space-time does imply some interesting

properties of physical objects. In particular, I wonder how the new constant l may reveal itself in experiment.

R. Some conceptions follow nearly automatically. Let us consider the group of linear homogeneous transformations of the eight objects $\boldsymbol{x}$, $\boldsymbol{d}$ characterizing quantized space-time (in natural units $c = l = 1$). The elements of the transformation matrix are complex numbers. Assume that the determinant of this matrix does not vanish. The above group contains two interesting sub-groups. One of them is the group of simultaneous transformations which do not mix the operators $\boldsymbol{x}$ and $\boldsymbol{d}$

$$\boldsymbol{x}'_\mu = a_{\mu\nu}\,\boldsymbol{x}_\nu, \qquad \boldsymbol{d}'_\mu = a_{\mu\nu}\,\boldsymbol{d}_\mu, \tag{6}$$

while the other mixes $\boldsymbol{x}$ and $\boldsymbol{d}$ but does not spoil their fourvector character

$$\boldsymbol{x}'_\mu = \alpha\boldsymbol{x}_\mu + \beta\boldsymbol{d}_\mu, \qquad \boldsymbol{d}'_\mu = \gamma\boldsymbol{x}_\mu + \delta\boldsymbol{d}_\mu. \tag{7}$$

The group (6) is the group of Lorentz transformations if we assume that the coefficients $a_{\mu\nu}$ satisfy the well known orthogonality relations. The Lorentz transformations preserve the commutation relations (5) and may be considered as a special case of canonical transformations. It is easily seen that a necessary and sufficient condition for (7) to be a canonical transformation is that the determinant be unity

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 1. \tag{8}$$

The coefficients $\alpha, \ldots \delta$ do not need to be real neither the matrix of these coefficients to be unitary or orthogonal. We see that the matrix of the transformation (7) contains (four times) the unimodular matrix

$$A = \begin{Vmatrix} \alpha & \beta \\ \gamma & \delta \end{Vmatrix}. \tag{9}$$

This has something to do with spinor theory. In order to exhibit the connection with spinors let us introduce the new notation

$$\boldsymbol{x}_\mu = {}_1u_\mu, \qquad l^2\boldsymbol{d}_\mu = {}_2u_\mu. \tag{10}$$

In terms of (10) the commutation relations (5') assume the elegant form

$$[{}_\varkappa u_\mu,\ {}_\varrho u_\nu] = l^2\,\gamma_{\varkappa\varrho}\,\delta_{\mu\nu} \qquad (\mu, \nu = 1, \ldots 4;\ \varkappa, \varrho = 1, 2), \tag{11}$$

Here $\gamma_{\varkappa\varrho}$ is the metric spinor

$$\gamma_{\varkappa\varrho} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}. \tag{12}$$

For a given μ, the pair of operators $\boldsymbol{x}_\mu$, $\boldsymbol{d}_\mu$, may be regarded as components of a spinor. This throws a new light upon the significance of the canonical transformations.

We may select from the group (7) or (9) the subgroup of unitary transformations or the subgroup of complex orthogonal transformations. These contain in turn a finite subgroup of reciprocal transformations whose matrices are powers of the matrix (12).

The transformations (6) and (7) commute with each other and have only one element in common: the identical transformation. Thus, we may consider a direct product of these two groups. Its representations are direct products of representations of the two groups.

Q. The point is what are the implications of these ideas for the theory of fields and elementary particles? In my opinion the idea of quantized space-time implies that the field functions are not functions of the classical quantities x_μ but functions of the elements of quantized space-time $\boldsymbol{x}_\mu$, $\boldsymbol{d}_\nu$:

$$\psi(\boldsymbol{x}_\mu, \boldsymbol{d}_\nu) \equiv \psi({}_\varkappa u_\mu). \tag{13}$$

This leads unavoidably to the conception of a bilocal field theory as, in the x-representation, the field functions become, in general, non-diagonal matrices

$$\psi = < x' \mid \psi \mid x'' > \equiv \psi(x', x'') \tag{14}$$

R. Moreover, another (perhaps no less important) consequence is the following: In the traditional field theory the field functions were characterized by the fact that they belonged to irreducible unitary representations of the Lorentz group (or the group of rotations). Now, in view of the existence of a larger group of transformations, such a characterization is obviously insufficient. It will be very natural to assume that the field functions (13) belong to irreducible unitary representations of the direct product of the groups (6) and (7). In other words, the field function (13) or (14) is a spinor (of a certain rank) with respect to the Lorentz transformations and simultaneously a spinor (of a certain rank) with respect to the unimodular group of canonical transformations (7):

$$\psi = {}_\varrho\psi_\alpha({}_\varkappa u_\mu), \tag{15}$$

where, in order to avoid confusion, the indices α, μ connected with the Lorentz transformations have been written to the right, as usually, while the indices ϱ, $\varkappa$ connected with the unimodular group (7) to the left. Thus, the bilocal field is a spinor in a twofold sense. The physical meaning of the index α is well known. It is closely connected with the spin of the elementary particles represented by the field function. The question arises as to the physical meaning of the other index ϱ. I have not the slightest idea what it means. Have you?

Q. Its role is not clear for me either. I can say only that, according to the order of representation, the field function represents a corresponding number of different particles or different states of polarization of a particle. It is not excluded that the differences between these particles (or states) will manifest themselves only when one introduces an interaction. As we do not possess a theory of interactions, I propose to introduce a provisional name which implies nothing except for a formal similarity with the ordinary spin and call it „isotopic spin". ϱ will be called the index of isotopic spin.

R. Well. Let us go over to the problem of field equations for bilocal field functions. In the local theory the requirements of linearity, relativistic covariance, and the requir-

ement that the order of equation be not higher than the second, led unambiguously to the Schrödinger-Gordon, or Dirac, or Kemmer equations. Now, we have to consider whether the requirement of covariance with respect to (linear) canonical transformations introduces any new limitations or not. Let us begin with recalling some well-known facts: in the theory of relativity one considers two types of scalar functions or operators: form-invariant scalars (e.g. the square of the four-distance, or the d'Alembert operator) and simply scalars determined by the transformation law

$$f'(x'_\mu) = f(x_\mu) \tag{16}$$

to be understood as an identity in x_μ (or x'_μ). Similarly, there exist form-invariant and not form-invariant vectors and tensors. An example of a form-invariant fourvector is the radius vector x_μ or the gradient, while an example of a non-form-invariant vector is a constant vector a_μ. Form-invariant quantities are isotropic whereas the others are covariant but not isotropic. The equations of motion (or the field equations) and the general expressions for physical quantities must be form-invariant since in this way we do justice to the isotropic character of space-time itself. On the other hand, the initial conditions or special solutions of the equations of motion (or the field equations) cannot be form-invariant since any concrete state of the physical system means a specific situation (e.g. the state of motion of a free particle specifies the direction of motion i.e. priviledges a direction in space-time). Similarly, one may distinguish two types of covariant quantities with respect to canonical transformations. However, we notice a difference in comparison with Lorentz transformations since the general expressions for such physical quantities as e.g. kinetic energy cannot be form-invariant as they denote specific characteristics from the point of view of the phase-space (p, q) in which the canonical transformations operate. Only still more general properties of the system such as equations in the Hamiltonian form or commutation relations may be form-invariant.

Q. Our aim is to investigate whether the idea of quantum space-time (bilocality) does or does not involve new possibilities of field equations for bilocal fields. Since, in the general framework of quantum theory based on the conception of quantized space-time, the local field theory is contained as a special case, let us realize first how a local field satisfying the Schrödinger-Gordon equation is to be described within the general framework of the theory. In this case we have

$$[_2 u_\mu, [_2 u_\mu, \psi]] = l^4 m^2 \psi, \tag{17'}$$

$$[_1 u_\mu, \psi] = 0 \quad _{(\mu = 1, \ldots, 4)}, \tag{17''}$$

(17') is the Schrödinger-Gordon equation whereas the four equations (17'') are the conditions of localizability. The dependence on l in (17') is only apparent as, due to (10), l^4 cancels on both sides. Obviously the set (17) appears rather clumsy contradicting the spirit if not the letter of space-time quantization. In contradistinction to (17) the following set

$$[_\varrho u_\mu, [_\sigma u_\mu, \psi]] = l^4 m^2 \delta_{\varrho\sigma} \psi \qquad _{(\varrho, \sigma = 1,2)} \tag{18}$$

seems more convincing. It is a set of four equations but only three of them are independent on account of the identity

$$[{}_{\varrho}u_{\mu}, [{}_{\sigma}u_{\nu}, f]] \equiv [{}_{\sigma}u_{\nu}, [{}_{\varrho}u_{\mu}, f]] \tag{19}$$

valid for any f. In contradistinction to (17) the constant l interferes in (18) essentially. It may be shown that the correspondence principle is satisfied since in the limit $l \to 0$ the set of (three independent) equations (18) degenerates into the set (17).

R. From the relativistic point of view the set (18) is superior to (17) since it involves only scalar and not vector conditions (scalars are more elementary than vectors). It is also superior from the point of view of canonical transformations. Both ${}_1u_{\mu}$ and ${}_2u_{\mu}$ appear in (18) in a symmetrical way. The set (18) is form-invariant with respect to the orthogonal subgroup of the group (9) (and of course also with respect to the reciprocal group). But it may be also made covariant under the whole unimodular group (9) or (7). To this end we have to replace the Kronecker symbol appearing to the right-hand side of (18) by a symmetric spinor reducing to the Kronecker symbol by means of a suitable unimodular transformation. I should like to ask you what are the solutions of (18)?

Q. In the x-representation the general solution may be written in the form

$$\psi(x, r) = \frac{1}{(2\pi)^2} \int d^4r \int d^4p\, e^{ipx}\, \delta(p^2 + m^2)\, \psi(p, r) \tag{20}$$

with

$$x = \frac{1}{\sqrt{2}}(x' + x''), \qquad r = \frac{1}{\sqrt{2}}(x' - x''). \tag{21}$$

The Fourier transform is

$$\psi(p, r) = N\, \delta(r\,p)\, \delta(r^2 + l^4 p^2)\, \varphi(p, r) \tag{22}$$

where N is a normalizing factor. The appearance of three delta functions in (20) and (22) corresponds to the fact that (18) involves three independent conditions for ψ. One of them, leading to $\delta(p^2 + m^2)$, may be regarded as the proper Schrödinger-Gordon equation while the remaining two may be regarded as supplementary conditions. They mean that the field quantity (22) vanishes unless

$$pr = 0,\ r^2 = -l^4 p^2 = l^4 m^2. \tag{23}$$

In the rest system ($\vec{p} = 0$) it is different from zero only for

$$r_0 = 0,\ |\vec{r}| = l^2 m. \tag{23'}$$

Thus, the particle is represented by the surface of a sphere with radius $l^2 m$ in its system of rest. Since the Fourier transform (22) involves Dirac delta functions, we may call it the „field distribution" to be distinguished from $\varphi(p, r)$ appearing to the right in (22) which may be called the „field skeleton". We may assume any values of the field skeleton beyond the domain defined by the constraints (23) since only its values on the surface (23) matter. For the points r_{μ} on the surface (23) we may develop φ in a

series of spherical harmonics. Thus we get

$$\varphi(p, r) = \sum_{n,m} \varphi_{nm}(p, r) \qquad \text{for } pr = 0, \qquad r^2 = -l^4 p^2, \tag{24}$$

where φ_{nm} are spherical harmonics transformed from the system of rest to the system of coordinates where the momentum of the particle is $\vec{p}$. Each of the spherical harmonics is multiplied by a function of $\vec{p}$ and sign p_0. These coefficients become annihilation or creation operators upon second quantization. Thus, we have to do here with a rotator model of an elementary particle. The spherical harmonics are eigenfunctions of the angular momentum of the rotator. But this means an internal motion of the particle so that the solutions φ_{nm} correspond respectively to particles with spin $\hbar n$ $(n + 1)$ its projection being $\hbar m$. Upon this internal motion there is superposed a translational motion of the particle as a whole. The variables x_μ are coordinates of the particle centre in space-time. The momentum and energy of the particle is p_μ. The state of the particle is characterized by the eigenvalues of five observables, e.g. momentum of the centre and angular momentum (together with its projection) of the internal motion. This is closely related to the fact that, besides the three translational degrees of freedom, we have to do with two further degrees of freedom describing the internal structure of the particle. Thus, we need five observables to form a complete set. This was point of view of first quantization. Upon second quantization the state is characterized by the occupation numbers N_{pnm}.

R. The above solutions may be represented in a different, though equivalent, way. We may regard the skeleton $\varphi(p,r)$ as a Fourier transform of the quantity

$$\varphi(x, r) = \frac{1}{(2\pi)^2} \int d^4 p \, e^{ipx} \, \delta(p^2 + m^2) \, \varphi(p, r) \tag{25}$$

subject to the constraints (23). Assuming that the bilocal field quantity is e.g. a scalar, we may write it in the form

$$\varphi(x, r) = \varphi(x) + r_\mu \varphi(x)_\mu + r_\mu r_\nu \varphi(x)_{\mu\nu} + \ldots, \tag{26}$$

where $\varphi(x)_{\mu\nu\ldots}$ are tensors of higher and higher ranks. In general these tensors would depend on r_μ^2 but on account of the second condition (23) this may be neglected. The first condition (23) states that the solution is independent of the component of r_μ in the direction of p_μ. This condition will be satisfied if we assume that the divergences of $\varphi(x)_{\mu\nu\ldots}$ vanish. Assuming that the traces of the tensors $\varphi(x)_{\mu\mu}$ vanish we secure that the higher terms do not contain parts already taken into account by lower terms. Finally, it is immediately seen that only symmetrical tensors come into account in (26). Therefore the bilocal field restricted by the auxiliary conditions (23) is equivalent to an infinite set of local fields (of Fierz) belonging to irreducible representations of the group of rotations. These fields describe particles with integer spin values. The formalism may be generalized for the case of a bilocal spinor field quantity $\varphi(xr)_a$. In this case we

would obtain all half-integer spin values corresponding to the set of local fields of the types $\varphi(x)_a$, $\varphi(x)_{a\mu}$, $\varphi(x)_{a\mu\nu}$,... The bilocal field quantity may be regarded as a generating function for an infinite set of local fields with higher spins. The fact that the bilocal field theory takes account of higher spins automatically, seems very satisfactory. In this way the theory avoids any distinction of a particularly chosen spin value and of a particular representation. It avoids specific and accidental distinctions and therefore assumes a more universal character. Since the physical space is characterized, above all, by the group of rotations, it seems very natural that all irreducible representations of this group should appear on equal footing. There should appear in the theory (at least) so many different bilocal field quantities that every irreducible (one or two-valued) representation of the group of rotations and reflections be taken into account. But these representations may appear several times due to the fact that, besides the characterization by spin and parity, the elementary particles may be characterized also by isotopic spin. All this will enable a general classification of the types of elementary particles. These particles may be grouped into families so that every family will be described by a bilocal field function.

Q. The opinion that the physical models investigated by quantum theory of fields are introduced as foreign elements, is to be revised in view of the bilocal field model. The very fact that a bilocal field function takes account automatically of all the representations of the group of rotations shows that there exists an intimate relation between space and the bilocal model. Also the field equations (18) fit tightly into the structure of quantized space-time. It seems to me that bilocal fields are not intruders in quantized space-time.

R. I agree that they fit much better to the notions of (quantized) space-time. What bothers me is the role of the mass parameter appearing in (18). This parameter is still a foreign element in the formalism and seems to be accidental. Such parameters are out of place in a fundamental theory.

Mass quantization

Q. One of the possible hypothesis about the origin of mass is that the mass should be a secondary effect brought about by the interaction (self mass). Another hypothesis consists in assuming that only a part of the mass is of the type of a self-mass while the remaining part, called mechanical mass, is of quite another origin.

R. The name „mechanical mass" explains nothing. Similarly the name „isotopic spin" explained nothing, but we introduced it since the bilocal theory offered a natural possibility for the existence of such a quality. Let us examine whether the bilocal theory offers also a natural possibility for the existence of a thing to be called mechanical mass. It seems to me that it does. More generally: any theory of elementary particles with a structure (i.e. with additional degrees of freedom) offers an interesting possibility of interpretation of the rest mass.

Q. Tell me about this possibility.

R. Relativity taught us that the total mass consists of two parts: the rest mass m_0 and the mass which might be called kinetical mass m_k

$$m_k = \frac{1}{c}\sqrt{\vec{p}^2 + m^2{}_0\, c^2} - m_0. \tag{27}$$

This suggests that perhaps the whole mass (together with the rest mass) is of a kinetic origin. By saying this I mean that, similarly as in (27), the mass m_k is a function of the momentum $\vec{p}$, and m_0 may be also a function of some momenta (assuming the value zero for vanishing momenta). Obviously, these may be only the momenta connected with the internal motion of the particle. In order to have an internal motion we must dispose of a model with an internal structure. The rotator model of the particle will suit our purpose very well. The idea of connecting the notion of rest mass with internal motion of the rotator seems attractive. We may define a scalar which, in the system of rest of the centre of the particle, is identical with the square of the angular momentum of the rotator, and identify this scalar with the rest mass (in natural units $c = l = h = 1$). In this way the rest mass is a manifestation of the internal movement of the particle. This idea is so attractive since it reduces the notion of rest mass (hithertoo regarded as primary) to still more elementary notions connected closely with space-time. If this idea is right, the name „kinetic mass" would be more suitable than „mechanical mass".

Q. The angular momentum of internal motion is quantized and possesses discrete eigenvalues. Therefore also the rest mass would assume discrete eigenvalues. I like this idea since it means a mass quantization. Am I right in saying that, in your opinion, the mass operator is proportional to the operator of the square of angular momentum of the internal motion in the system of rest of the particle?

R. Not exactly. It may be a function of this operator. However, I do not insist that it must be necessarily a function of the squared angular momentum. I mentioned it rather as an example although I do not see, at the moment, any other possibility.

Q. If we replace in (18) the mass parameter by an operator, we must take care for this operator to be consistent with the constraints. Going over to the x-representation and introducing the momenta p', p'' conjugate to x', x''

$$p'_\mu = -ih\frac{\partial}{\partial x'_\mu}, \qquad p''_\mu = -ih\frac{\partial}{\partial x''_\mu} \tag{28}$$

or, in terms of the variables (21),

$$p_\mu = -ih\frac{\partial}{\partial x_\mu}, \qquad k_\mu = -ih\frac{\partial}{\partial r_\mu}, \tag{28'}$$

the operator of the rest mass cannot depend on $\vec{p}$ but only on p_4 and upon those k_μ which are consistent with the constraints (23). Thus, in the rest system of reference,

it cannot depend upon k_4 and k_r. This suggests that it is in fact a function of the operator of angular momentum of the internal motion only. The point is to find out which function it might be.

R. Perhaps our task will be made easier if we forget for a moment about the constraints. In the theory of the bipoint there should exist a relation between the momenta analogous to the famous relation $p_\mu^2 + m_0^2 = 0$. Now, we have a pair of points with momenta p' and p''. The new relation must be symmetrical in p' and p'' and should not contain any parameter, since the rest mass has to follow from the formalism but not to be introduced from the outside. The simplest relation of this type is $(p' + p'')^2 = 0$, but this would miss the point as, in terms of the variables x, r and their conjugate momenta it becomes $p_\mu^2 = 0$, which means a vanishing rest mass. The next possibility is

$$p'_\mu p'_\mu + p''_\mu p''_\mu = 0 \tag{29}$$

or, in terms of the momenta p, k conjugate to x, r

$$p_\mu^2 + k_\mu^2 = 0. \tag{29'}$$

In this last formula there appears a term k_μ^2 which could play the role of the rest mass. In the quantum theory both p_μ and k_μ are operators (28') and the relation (29') becomes a condition upon the field function

$$(p_\mu^2 + k_\mu^2)\, \psi\,(x, r) = 0. \tag{30}$$

(30) may be regarded as a generalized Schrödinger-Gordon equation. However, this equation is not consistent with the constraints. We may reconciliate it with the constraints by introducing suitable reaction forces, so that (30) assumes a new form

$$(p_\mu^2 + k_\mu^2)\, \psi\,(x, r) = R\, \psi\,(x, r). \tag{30'}$$

The operator R has to take account of the constraints (23), i. e. must be of such a form that the displacements inconsistent with the constraints drop out from (30'). In the rest system of reference k_4 and k_r are inconsistent and have to be eliminated. In this way we get

$$m^2 = k^2{}_\mu - R = \frac{S^2}{r_\mu^2}, \tag{31}$$

where S^2 is the (relativistically generalized) operator of angular momentum of the internal motion. According to the constraint $r_\mu^2 = l^4\, m^2$ (in the units $c = h = 1$), we find the rest mass to be proportional to the fourth root of S^2, and the mass eigenvalues are

$$m_n = \frac{1}{l} \sqrt[4]{n(n+1)} \tag{32}$$

Q. This shows that rest masses tend to infinity for $l \to 0$. Thus, the local field theory constitutes a singular case. Your deduction of the mass operator seems very elegant

but I do not care so much for elegance and think it advisable to assume a little less elegant but more general starting point, namely

$$p_\mu^2 + Ck_\mu^2 = 0. \tag{33}$$

The dimensionless constant C may assume different values for different families of elementary particles while l may preserve a universal value.

R. It is difficult to say in advance whether it will be necessary to introduce a new constant C by (33). Similar considerations may be applied to the cases of Dirac and Kemmer equations. The mass operator is expected to be suitably modified in those cases.

Q. What troubles me, is the fact that the rest mass operator seems to be not covariant with respect to canonical transformations.

R. I think it is covariant *ex definitione*, although it is not form-invariant. We encounter here a similar situation as e. g. for the Hamiltonian in classical mechanics. The Hamiltonian is *ex definitione* invariant but cannot be form-invariant. Our rest mass operator is even not form-invariant with respect to Lorentz transformations, and assumes a simple form only in the system of rest of the particle.

Q. Let us resume our results. The traditional local field theory was based on a point model of an elementary particle; the bilocal field theory may be regarded as a natural generalization of the former for the case of a bipoint model of an elementary particle. The point model is a classical model, whereas the bipoint model may be regarded as a genuine quantum model, being closely connected with the idea of space-time quantization, implying that every function is a matrix, i. e. a function of a pair of points. This pair of points, subject to constraints (covariant with respect to canonical transformations) constitutes the bipoint, or more exactly, the rotator model. The rest mass of the bipoint may consist of two parts: a field mass and a mechanical, or rather a kinetic mass. The latter is a manifestation of the internal motion of the rotator. It is plausible that, for higher states of internal motion, the field mass will be negligeable in comparison with the kinetic mass. Also the spin of the particle consists of two parts: it is a combination of (*i*) the „eigen-spin" determined by the transformation character of the bilocal field function, and (*ii*) the angular momentum of the internal motion. In the particular case of a scalar (or pseudoscalar) bilocal field quantity the „eigen-spin" is zero. The bilocal field theory offers a natural possibility for introducing the notion of isotopic spin. This is connected with the fact that the bilocal quantities belong not only to the irreducible representations of the Lorentz group but also to the irreducible representations of the group of linear canonical transformations. Isotopic spin might be called also „canonical spin".

R. Mechanical notions have been reduced to kinematic or geometric conceptions whereas geometry itself acquired a generalized meaning and became an abstract algebra. I would like so much to survive until the experimental results will show unambigously which part of our speculations corresponds to reality.

КРАТКОЕ СОДЕРЖАНИЕ

Г. Райский. *Дисскусия о билокальности.*

Ситуация понятия в современной теоретической физике подвергнута дисскусии. Операторы положения и бесконечно малых перемещений x, d создают алгебру, составляющую отвлечённое обобщение понятия време-пространства. Операторное разнообразие x, d составляет квантовое време-пространство и является исходной точкой для билокальной теории. Канонические трансформации этих операторов должны играть очень существенную роль. Линейная подгруппа группы канонических трансформаций унимодулярна, что позволяет на очень простое и естественное физическое толкование изоспина (спина изобарного, изостопного). Продисскутирован тоже вопрос покойной массы. Состоит она из двух частей: из массы кинетической (механической) и массы полевой. Первая из них связана с состоянием внутреннего движения бипунктовой частицы. Для более высоких степеней внутреннего движения можно вероятно пренебречь массой полевой, как меньшей, по сравнению с массой кинетической.

TWO-CENTER INTEGRALS FOR BODY-CENTERED IRON USING ATOMIC FUNCTIONS WITH EXCHANGE

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Two-center integrals for metallic body-centered iron have been calculated using the new atomic wave functions calculated with exchange by J. H. Wood. The integrals turn out to be about two times smaller than those based on the wave functions without exchange.

For the tight binding approximation, the calculation of the matrix elements presents serious difficulties. They should, in principle, be calculated as linear combinations of three-center integrals.

One way out has been proposed by Fletcher and Wohlfarth (1951) in their work on the $3d$ electron band in nickel. They have computed the necessary energy integrals as two-center integrals based on appropriate atomic self-consistent functions calculated with exchange. Since the mean radius of the $3d$ function for the important transition metals is small in comparison with the interatomic distance it is hoped, in this way, to obtain a reasonable approximation for the values of the energy integrals.

It is known that $3d$ functions in transition elements are very sensitive to exchange effects. For iron, the only calculations of self-consistent functions to be found in the literature, namely, Manning and Goldberg (1938) did not include the exchange. Recently, J. H. Wood (1955) of MIT has computed the self-consistent functions of the Fe atom taking into account the exchange effects by using the free electron approximation proposed by Slater (1951). The normal state configuration of the Fe atom has been taken as five $3d1$ electrons (spin up) and one $3d2$ electron (spin down). The functions computed by Wood are much more compact than the older functions of Manning and Goldberg. The maximum of the $3d1$ function of Wood is at $r = 0.68$, of the $3d2$ function, at $r = 0.7$, and of the $3d$ function of Manning and Goldberg, at about $r = 0.75$ Bohr units.

The $3d1$ function of Wood has now been used to calculate the energy two-center integrals for metallic body-centered iron. The numerical values of the normalized $P_N(3d1; r)$ function of Wood have been fitted by the analytical function

$$P_A(r) = r^3(2.815\, e^{-2.1r} + 81.97\, e^{-5.1r}) \tag{1}$$

This function fits the figures of Wood fairly accurately, a discrepancy of 40 to 64×10^{-3} occuring from $r = 0.12$ to $r = 0.35$, while for other ranges of r, the discrepancy remains less than 36×10^{-3} (See table). The normalization integral is

$$\int_0^\infty P_A^2(r)\, dr = 0.999896.$$

For the potential energy, the expression used was

$$U(r) = -(1 + 25\, e^{-3r})/r \qquad (2)$$

though it is fully realized that this is not well founded.

TABLE

of $3d1$ function of iron from unpublished tables of J. H. Wood

r	$P_N(3d1;r)$	$P_A(r)$	$(P_A - P_N) \times 10^3$	r	$P_N(3d1;r)$	$P_A(r)$	$(P_A - P_N) \times 10^3$
0.02	0.0012	0.0006	— 0.6	0.68	0.9824	1.0159	33.4
0.04	0.0079	0.0044	— 3.4	0.69	0.9823	1.015	32.7
0.06	0.0227	0.0136	— 9.1	0.70	0.9818	1.0136	31.8
0.08	0.0462	0.0291	—17				
0.10	0.0777	0.0515	—26.2	0.75	0.9746	1.0003	25.8
0.12	0.1163	0.0806	—35.7	0.80	0.9604	0.9782	17.8
0.14	0.1604	0.1159	—44.5	0.90	0.9174	0.9167	— 0.7
0.16	0.2088	0.1567	—52.1	1.0	0.8621	0.8445	—17.6
0.18	0.2601	0.2021	—58	1.1	0.801	0.7713	—29.6
0.20	0.3131	0.2513	—61.8	1.2	0.7381	0.7028	—35.3
0.22	0.3667	0.3031	—63.6	1.3	0.6762	0.6411	—35.1
0.24	0.4201	0.3567	—63.4	1.4	0.6168	0.5867	—30.1
0.26	0.4725	0.4112	—61.2				
0.28	0.5232	0.4658	—57.4	1.6	0.5087	0.4965	—12.2
0.30	0.5719	0.5197	—52.2	1.8	0.4166	0.424	7.4
				2.0	0.3398	0.3621	22.3
0.35	0.6826	0.6476	—35	2.2	0.2765	0.307	30.5
0.40	0.7753	0.7599	—15.4	2.4	0.2246	0.2574	32.8
0.45	0.8492	0.8523	3.2	2.6	0.1827	0.213	30.3
0.50	0.905	0.9232	18.2	3.4	0.0786	0.0887	10.1
0.55	0.9442	0·9727	28.6	4.2	0.0334	0.0308	— 2.6
0.60	0.9686	1.0026	34	5.0	0.0139	0.0097	— 4.2
0.65	0.9805	1.0153	34.8	5.8	0.0052	0.0028	— 2.4
0.66	0.9815	1.0161	34.5	6.6	0.0007	0.0008	0.1
0.67	0.9822	1.0162	34				

The lattice parameters of iron have been taken from the latest measurements of Basinski, Hume-Rothery and Sutton (1955). The distance to the nearest neighbours was taken as $d = 2.4772\ kX$, to the next nearest, $a = 2.8604\ kX$. In order to convert these data to Ångström units and thence to Bohr units, the values from the table

of Cohen and Du Mond (1953) have been taken. Thus, $d = 2.4823$ Å, $a = 2.8663$ Å and the values $d = 4.690\,941\,99$ and $a = 5.416\,587\,46$ atomic Bohr units were used in the present calculations. The atomic unit of energy was taken as 27.21 eV.

As in the previous paper published by the author (Acta phys. Polon., 1955), referred to, hereafter, as I, in computing the integrals, use has been made of all the

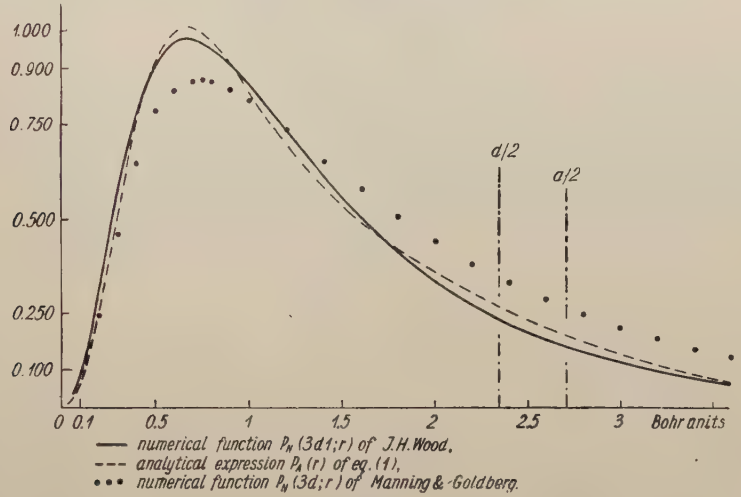

Fig. 1

approximations used by Fletcher (1952). Apparently, the errors introduced by these approximations are considerable and constitute a serious shortcoming of the present work.

Values obtained for the integrals of σ-, π-, and δ-type symmetry, computed using (1) and (2) are:

for the nearest neighbours

$$\begin{aligned}(dd\,\sigma)_1 &= -0.411\,296 \text{ eV}\\ (dd\,\pi)_1 &= 0.220\,982 \text{ eV}\\ (dd\,\delta)_1 &= -0.033\,215 \text{ eV}\end{aligned} \tag{3}$$

for the next nearest neighbours

$$\begin{aligned}(dd\,\sigma)_2 &= -0.168\,693 \text{ eV}\\ (dd\,\pi)_2 &= 0.074\,512 \text{ eV}\\ (dd\,\delta)_2 &= -0.009\,405 \text{ eV}\end{aligned} \tag{3'}$$

The last two decimal places given here may be not significant and they are included only to avoid truncation.

By comparing the present values with those caluculated using atomic functions without exchange, it can be seen that they are now considerably diminished. Denoting the values given in paper I by the index I, we obtain

$$(dd\,\sigma)_1^I/(dd\,\sigma)_1 = 1.7$$
$$(dd\,\pi)_1^I/(dd\,\pi)_1 = 1.9 \qquad (4)$$
$$(dd\,\delta)_1^I/(dd\,\delta)_1 = 2.1$$

$$(dd\,\sigma)_2^I/(dd\,\sigma)_2 = 1.9$$
$$(dd\,\pi)_2^I/(dd\,\pi)_2 = 2.1 \qquad (4')$$
$$(dd\,\delta)_2^I/(dd\,\delta)_2 = 2.3$$

It must be said that the distances of the neighbours adopted in I were a bit smaller than those adopted here because in I the Ångström unit has been taken as equal to the kX unit. It is estimated that this difference of about 2 promille in the distances of neighbours will decrease tha values of I by about 1 percent.

In conclusion, it can be said that atomic functions calculated with exchange give more reasonable results in computing the energy integrals in metals. In view of

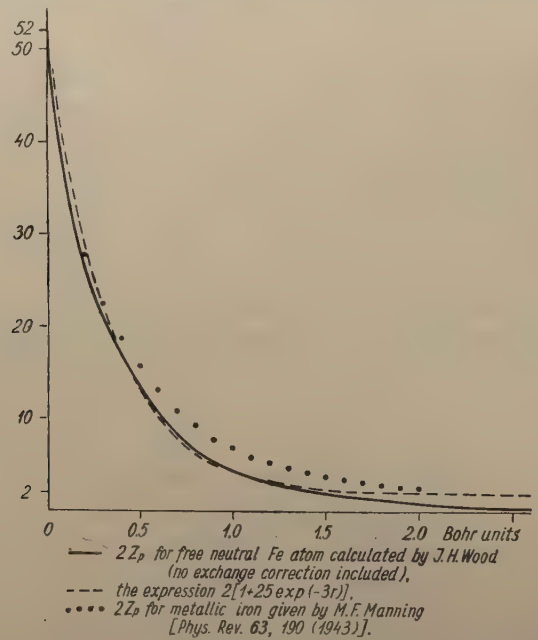

Fig. 2

the very recent work of Callaway (1955) on Fe it seems that the present values are still too great, whereas in view of the results of Howarth (1955) on Cu the present values are quite meaningful.

The interesting work of Frank Stern (1955) has revealed that the configuration adopted for $3d$ electrons of iron has a very great effect on the density of atomic $3d$ electrons. This is a further reason why the quantitative grasp of tight binding integrals is really a difficult task.

I am very grateful to Dr. J. H. Wood for generously sending me his unpublished tables of atomic functions of Fe. I am also greatly indebted to Dr. G. C. Fletcher for his kind correspondence and for communicating to me the simple method for estimating the variation of energy integrals with interatomic distance. I also thank Professor J. Callaway for sending me his unpublished papers and Professor J. Slater for sending his paper on the tight binding method (1954).

КРАТКОЕ СОДЕРЖАНИЕ

M. Суффгинский. *Бицентровые интегралы для пространственно-центрового железа при употреблении атомных функций с обменом.*

Исчислены бицентровые интегралы для металлигеского, пространственно-центрового железа, пользуясь новыми атомными волновыми функциями, вычисленными, учитывая обмен J. H. Wooda. Интегралы эти приходятся два раза меньше вычисленных без учёта обмена.

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ON THE MASS SPECTRA FOR BOSONS

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Two different mass operators consistent with the bilocal theory are constructed for bosons. One of them seems to apply to the pion family, the other (constructed with the aid of Duffin-Kemmer matrices) seems to apply to the K-meson family (Θ^0-family).

Denoting the coordinates of the particle centre by x_μ and the coordinates of the internal structure by r_μ, and introducing the notation

$$p_\mu = -\frac{\partial}{\partial x_\mu}, \qquad k_\mu = -\frac{\partial}{\partial r_\mu}, \tag{1}$$

the bilocal field equations for bosons with mass m (in units $\hbar = c = l = 1$) are

$$(p_\mu p_\mu + m^2)\,\psi(x, r) = p_\mu r_\mu \psi(x, r) = (r_\mu r_\mu - l^4 m^2)\,\psi(x, r) = 0. \tag{2}$$

It is convenient to go over to the momentum representation

$$\psi(x, r) = \frac{1}{(2\pi)^2} \int d^4 p\, e^{ipx}\, \psi(p, r) \tag{3}$$

so that equations (2) become simple algebraic relations for $\psi(p, r)$, and their solution is

$$\psi(p, r) = \delta(p_\mu p_\mu + m^2)\,\delta(p_\mu r_\mu)\,\delta(r_\mu r_\mu - l^4 m^2)\,\varphi(p, r). \tag{4}$$

$\varphi(p, r)$ is still an arbitrary function of $\vec{p}$, of sign p_0, and of two of the internal structure variables. In the system of coordinates where $\vec{p} = 0$ the internal structure variables are the angles φ, ϑ on a sphere of radius $|\vec{r}| = l^2 m$.

In previous papers by one of the authors (Rayski 1955) the field equations (2) have been generalized by replacing the square of the rest mass parameter m^2 by an operator M^2. This operator (a scalar with respect to Lorentz transformations) must be consistent with the field equations (2). To secure the consistency M^2 must commute with the arguments of the δ — functions appearing in (4), so that the mass eigenequation

$$M^2 \psi = m^2 \psi \tag{5}$$

may be regarded as a conditions upon the function $\varphi(p, r)$ appearing on the right hand side of (4)

$$M^2\varphi = m^2\varphi. \tag{5'}$$

Thus, the square of the mass operator must commute with $p_\mu r_\mu$ and $r_\mu r_\mu$ or, in the reference system in which $\vec{p} = 0$, it must commute with r_4 and $|\vec{r}|$, i. e. it cannot contain k_4 and k_r.

Since the bipoint model is a rotator, the suggestion arises that M^2 be identified with the angular part of the d'Alembert operator for the internal structure variables

$$a^4 M^2 = \frac{S^2}{r^2}_{[\vec{p}=0]}, \tag{6}$$

where a^4 is a factor of proportionality, and S^2 is the angular momentum operator with the eigenvalues $n(n+1)$. On account of the appearance of the δ — functions in (4) r^2 may be replaced in (6) by $l^4 M^2$ whence the mass eigenvalues are

$$m_n = \frac{1}{al}\sqrt[4]{n(n+1)} \quad , n = 0, 1, ... \tag{7}$$

The solutions of the mass eigenequation (5) are spherical harmonics. In spite of the fact that the bilocal field quantity is scalar or ps-scalar, the particles with mass m_n possess an angular momentum of internal motion, i. e. a spin n. The rest mass may be interpreted as a manifestation of the internal motion of the particle. The mass operator (6) may also be written

$$M^2 = \frac{1}{a^4}(-\Box_r + R) = \frac{1}{a^4}(k_\mu k_\mu + R), \tag{8}$$

where R is an invariant hermitian operator which eliminates those of the $k'_\mu s$ not consistent with equations (2). This operator assumes a particularly simple form in the rest system of the particle

$$R_{[\vec{p}=0]} = \left(\frac{\partial^2}{\partial r_4^2} + \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) \tag{9}$$

and may be understood to a result from the existence of reaction forces arising by the constraints $p_\mu r_\mu = 0$, $r_\mu r_\mu + l^4 p_\mu p_\mu = 0$. With the aid of (8) the Schrödinger-Gordon equation may also be written in the form

$$\left(\Box_x + \frac{1}{a^4}\Box_r\right)\psi = R\psi. \tag{10}$$

The aim of the present paper is to point out another interesting alternative for a rest mass operator for bosons. This alternative mass operator may be constructed with the aid of Duffin-Kemmer matrices satisfying the well known commutation relations

$$\beta_\mu \beta_\nu \beta_\varrho + \beta_\varrho \beta_\nu \beta_\mu = \beta_\mu \delta_{\nu\varrho} + \beta_\varrho \delta_{\nu\mu}. \tag{11}$$

Let us consider the formal „root" of the operator $k_\mu k_\mu$

$$K = \beta_\mu k_\mu = -i \beta_\mu \frac{\partial}{\partial r_\mu} \tag{12}$$

and eliminate in it the derivatives inconsistent with the equations (2) by introducing suitable reaction forces (i. e. a hermitian scalar R)

$$N = K + R. \tag{13}$$

K may be split into a time-like, a radial, and an angular part

$$K = -i \left[\beta_4 \frac{\partial}{\partial r_4} + \beta_r \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) + \frac{\Omega}{r} \right] \tag{14}$$

where

$$\Omega = -\beta_r + \beta_\vartheta \frac{\partial}{\partial \vartheta} + \frac{1}{\sin \vartheta} \beta_\varphi \frac{\partial}{\partial \varphi} \tag{15}$$

(ϑ, φ being the usual spherical coordinates $x = r \sin \vartheta \cos \varphi$, $y = r \sin \vartheta \sin \varphi$, $z = r \cos \vartheta$ and β_ϑ, β_φ denoting the components of β in the directions of meridian and parallel circles respectively). The role of R in (13) is to eliminate (in the rest system $\vec{p} = 0$) the time-like and radial parts. Thus, R must be a relativistic generalization of

$$R_{[\vec{p}=0]} = i \left[\beta_4 \frac{\partial}{\partial r_4} + \beta_r \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \right]. \tag{16}$$

A new form of the square of the mass operator may be defined by means of

$$M^2 = \frac{1}{a^4} N^2 \tag{17}$$

In other words, the square of the mass operator is a scalar which, in the rest system of the particle, becomes identical with the square of the operator $\frac{-i\Omega}{a^2 r}$, subject to the condition $r = l^2 M$. Hence, the mass eigenvalues m are connected with the eigenvalues Ω' of the operator Ω by

$$m = \frac{1}{al} \sqrt[4]{-\Omega'^2}. \tag{18}$$

Let us compute the eigenvalues and eigenfunctions of the operator $-i\Omega$ in a five row and column representation of the Duffin-Kemmer matrices. First of all we shall demonstrate that a zero eigenvalue does not exist. If we assume that Ω possesses a zero eigenvalue, at least one of the field components $\psi_a^{(0)}$, forming eigensolutions to this eigenvalue, does not vanish identically. We assume that this is just $\psi_5^{(0)}$. This does not constitute any restriction on the generality since we are allowed to label the field components arbitrarily. Since we are allowed to assume any represen-

tation of the β — matrices, we may assume, for the case $\psi_5^{(0)} \neq 0$, the representation given by Kemmer (1939)

$$\beta_1 = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{vmatrix}, \quad \beta_2 = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{vmatrix}, \quad \beta_3 = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \end{vmatrix} \tag{19}$$

whence

$$\beta_\vartheta = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cos\vartheta\cos\varphi \\ \cdot & \cdot & \cdot & \cdot & \cos\vartheta\sin\varphi \\ \cdot & \cdot & \cdot & \cdot & -\sin\vartheta \\ \cdot & \cos\vartheta\cos\varphi & \cos\vartheta\sin\varphi & -\sin\vartheta & \cdot \end{vmatrix},$$

$$\beta_\varphi = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -\sin\varphi \\ \cdot & \cdot & \cdot & \cdot & \cos\varphi \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -\sin\varphi & \cos\varphi & \cdot & \cdot \end{vmatrix},$$

$$\beta_r = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sin\vartheta\cos\varphi \\ \cdot & \cdot & \cdot & \cdot & \sin\vartheta\sin\varphi \\ \cdot & \cdot & \cdot & \cdot & \cos\vartheta \\ \cdot & \sin\vartheta\cos\varphi & \sin\vartheta\sin\varphi & \cos\vartheta & \cdot \end{vmatrix} \tag{20}$$

(dots denote zeros).

In this representation, the eigenequation

$$-i\Omega\psi = -i\Omega'\varphi \tag{21}$$

assumes the form

$$i\Omega'\psi_1 = 0 \tag{22a}$$
$$A\psi_2 + B\psi_3 + C\psi_4 = i\Omega'\psi_5 \tag{22b}$$
$$A\psi_5 = i\Omega'\psi_2 \tag{22c}$$
$$B\psi_5 = i\Omega'\psi_3 \tag{22d}$$
$$C\psi_5 = i\Omega'\psi_4 \tag{22e}$$

where

$$A = -i\left(\sin\vartheta\cos\varphi - \cos\vartheta\cos\varphi\frac{\partial}{\partial\vartheta} + \frac{\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right)$$
$$B = -i\left(\sin\vartheta\sin\varphi - \cos\vartheta\sin\varphi\frac{\partial}{\partial\vartheta} - \frac{\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right) \tag{23}$$
$$C = -i\left(\cos\vartheta + \sin\vartheta\frac{\partial}{\partial\vartheta}\right).$$

One can easily verify (e. g. coming back to the Cartesian coordinates) that A, B, C are components of a vector operator $-\frac{i}{r}\{\vec{r}+\vec{r}\times(\vec{r}\times\vec{r})\}$ whose square is related to the square of the angular momentum operator S^2 by

$$A^2+B^2+C^2=1+S^2. \tag{24}$$

Putting $\Omega'=0$ we should have

$$A\psi_5^{(0)}=B\psi_5^{(0)}=C\psi_5^{(0)}=0, \tag{25}$$

whence $\psi_5^{(0)}\equiv 0$ contrary to the former assumption. Therefore the eigenvalue $\Omega'=0$ does not exist. Equation (22a) then gives $\psi_1\equiv 0$. Multiplying (22b) by $\Omega'\neq 0$, and using (22c, d,e) one obtains

$$(A^2+B^2+C^2)\,\psi_5=-\Omega'^2\psi_5, \tag{26}$$

or with the help of (24)

$$S^2\psi_5=-(1+\Omega'^2)\,\psi_5. \tag{27}$$

This yields for the eigenvalues of $-\Omega^2$

$$-\Omega'^2=1+n\,(n+1),\quad n=0,1,... \tag{28}$$

The eigenfunctions $\psi_5^{(n)}$ are spherical harmonics. The remaining non-vanishing functions $\psi_2^{(n)}$, $\psi_3^{(n)}$, $\psi_4^{(n)}$ can be expressed easily from (22*c*, *d*, *e*) by $\psi_5^{(n)}$ and its derivatives.

Putting (28) into the mass equation (18) one obtains

$$m_n=\frac{1}{al}\sqrt[4]{1+n\,(n+1)}\ ,\ n=0,1,\ldots \tag{29}$$

This mass spectrum differs from (7) which was obtained by the first procedure where the mass operator was introduced for boson fields. In both cases the particle with mass m_n possesses a spin $\hbar n$.

The suggestion arises that the mass spectrum (7) be related to the pion family and the mass spectrum (29), to the K-meson family. In this way we account for the comparatively small value of the pion mass (its entire mass 273 being only a field mass), and for the comparatively large value 965 of the K-meson mass. Of course, a part of the mass must be a self-mass. Assuming this part to be small or approximatively proportional to the mechanical mass, we may compute the mass values m_n for $n=1, 2, \ldots$ from (29) by identifying m_0 with the total mass of the K-meson. In this way we obtain from $m_0=965$

$$m_1=1270,\ m_2=1570,\ \text{etc...}$$

Although some authors (Daniel et al. 1954, Husain et al. 1955, Buchanan 1954) reported to have observed mesons with a mass of about 1270, their results are not conclusive and can be a statistical fluctuation. Probably the heavier members of the K-boson family are very short-lived and decay immediately with the emission of a γ or a pion. In connection with this possibility we would like to call attention to another

suggested interpretation of the results of the Cosmotron Group (Fowler et al. 1955). They interpret their cases A, B, G, I as a two-step reaction where the well known K°_{o}-meson (namely Θ) is produced together with Σ° and decays immediately into $\Lambda^{\circ} + \gamma$. But it is also possible to interpret them in the reversed sense: a heavier meson $K^{\circ}{}_{1}$ is produced together with Λ° and decays immediately into the usual $K^{\circ}_{o} = \Theta^{\circ}$ ($K^{\circ}{}_{1} \rightarrow K^{\circ}{}_{0} + \gamma$). Under this assumption the masses of the K-mesons in cases A, B, G, I are 650, 620, 670, and 625 MeV respectively. These values are consistent with each other within the experimental errors ($\pm$ 70 MeV). The mean value of the four cases is 641 MeV or 1260 m_e. This value agrees very well with our 1270 for the next member of the K-meson family. This agreement is striking enough and worth considering but is still far from constituting an evidence for the existence of K-mesons with mass above 1200 m_e.

КРАТКОЕ СОДЕРЖАНИЕ

В. Ханус и Г. Райский. *О спектре бозоновых масс*

Сооружены, согласно требованиям билокальной теории поля, два, разнящиеся между собою, массовые оператора для бозонов. Кажется, что один из них относится к семейству мезонов π, другой, (сконструированный при помощи матрицы Duffina-Kemmera), к семейству мезонов К (семейства Θ°).

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A VARIATIONAL PRINCIPLE FOR BILOCAL FIELD THEORY

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The generalized Schrödinger-Gordon-Fock equations of the bilocal field theory are derived as the Euler-Lagrange equations of a suitably formulated variational principle

Bilocal field equations are Euler-Lagrange equations derivable from a generalized variational principle where the action integral is a scalar with respect to Lorentz transformations but a symmetric second rank spinor with respect to canonical transformations (iso-spinor).

In the bilocal field theory a generalized conception of field quantity

$$\psi({}_{\varrho}u_{\mu}) \qquad \varrho = 1,2; \qquad \mu = 1,\ldots,4 \tag{1}$$

satisfying a generalized Schrödinger-Gordon-Fock equation

$$[{}_{\varrho}u_{\mu}, [{}_{\sigma}u_{\mu}, \psi]] = l^4 M^2 \delta_{\varrho\sigma} \psi \qquad \varrho, \sigma = 1,2 \tag{2}$$

is investigated. The symbols ${}_{\varrho}u_{\mu}$ denote operators satisfying the following commutation relations

$$[{}_{\varrho}u_{\mu}, {}_{\sigma}u_{\nu}] = l^2 \gamma\delta_{\varrho\sigma} \delta_{\mu\nu} \qquad \varrho,\sigma = 1,2; \quad \mu,\nu = 1,\ldots,4. \tag{3}$$

where

$$\left\| \gamma_{\varrho\sigma} \right\| = \left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\|, \tag{4}$$

l denotes a fundamental length, and M^2 denotes the square of the mass operator (in units $\hbar = c = 1$). For the present purpose it is not necessary to specify this operator. For reasons of simplicity let us assume that we are working in the mass representation so that M^2 may be regarded as a c-number denoting an eigenvalue of the mass operator.

The relations (1) — (4) are covariant with respect to the Lorentz group of transformations

$${}_{\varrho}u'_{\mu} = a_{\mu\nu}\, {}_{\varrho}u_{\nu} \tag{5}$$

and with respect to the group of transformations

$${}_{\varrho}u'_{\mu} = {}_{\varrho\sigma}\alpha\, {}_{\sigma}u_{\mu} \tag{6}$$

where the matrices with two rows and colums

$$|| {}_{\varrho\sigma}\alpha || \tag{7}$$

are complex orthogonal. By generalizing suitably the Kronecker symbol $\delta_{\varrho\sigma}$ appearing in (2) the restriction that (7) is complex orthogonal may be abandoned, and covariance with respect to the broader group of unimodular transformations

$$\mathrm{Det}\ {}_{\varrho\sigma}\alpha = 1 \tag{8}$$

may be assumed. Thus, ${}_{\varrho}u_{\mu}$ is a four-vector with respect to (5) and a spinor with respect to (6) and (8). Also the field quantity may be regarded as a spinor (of any rank) with respect to both types of transformations ${}_{\varkappa}\psi_{a}(u)$. In order to avoid confusions the indices connected with the Lorentz group are written as usual on the right, while the indices connected with the group of transformations (6), on the left. The index $\varkappa$ denoting the transformation character of the field quantity with respect to (6) and (8) may be connected with the notion of isobaric*) spin (compare Rayski 1956).

In the present paper we are concerned with the problem of generalization of the variational principle so that equations (2) be derivable as Euler-Lagrange equations. Equations (2) differ in a twofold respect from the traditional field equations. First of all, they are abstract (operator) equations, secondly they constitute a set of four equations for a single field quantity (three of them being independent). The first of the two above mentioned differences is not essential since we may go over to a representation and convert the operator equations into ordinary equations. The other difference is more essential and distinguishes the bilocal field theory from the traditional (local) field theory, where the number of field equations was equal to the number of field quantities. The most suggested way out of this difficulty is to regard only one of the equations (2) as a „true" field equation, and the remaining as supplementary conditions. This method was followed by Bloch (1950). However, the set (2) is so symmetric that such a procedure constitutes an unjustifiable discrimination against three of the equations (2) thus violating the spirit of bilocality. Therefore, we propose a generalization of the Hamilton principle. This allows us to treat the four equations (2) on equal footing.

We define a generalized Langrangean operator

$${}_{\varrho\sigma}L = [{}_{\varrho}u_{\mu}, \psi^{*}]\,[{}_{\varrho}u_{\mu}, \psi] + l^{4} M^{2} \delta_{\varrho\sigma}\, \psi^{*}\psi \tag{9}$$

and an action integral

$${}_{\varrho\sigma}W = \int d^{4}\xi' < \xi' |_{\varrho\sigma} L \,|\, \xi|'' >, \tag{10}$$

where ξ'_{μ} denote eigenvalues of a complete set of observables in the Hilbert space spanned by the operators ${}_{\varrho}u_{\mu}$. The generalized Hamilton principle is

$$\delta\,({}_{\varrho\sigma}W) = 0 \tag{11}$$

As $\varrho, \sigma = 1, 2$, we have to do here with four variational conditions instead of a single

*) Isobaric spin or isospin are names for isotopic spin.

condition of the traditional theory. The four variational conditions (11) may be shown to be consistent with each other. If parameter M is to be replaced by an operator $\boldsymbol{M}$ care must be taken to preserve this consistency.

With the aid of (9) the variational conditions may be written explicitly

$$\delta \int d^4\xi'\, d^4\xi''\, \{<\xi' \mid [{}_\varrho u_\mu, \psi^*] \mid \xi'' > <\xi'' \mid [{}_\sigma u_\mu, \psi] \mid \xi' > + l^4 M^2 \delta_{\varrho\sigma} <\xi' \mid \psi^* \mid \xi'' > <\xi'' \mid \psi \mid \xi' >\} = 0. \tag{12}$$

The commutation relations (3) may be realized by putting e.g.

$${}_1u_\mu = x_\mu, \quad {}_2u_\mu = l^2 d_\mu \tag{13}$$

where x_μ and d_μ are the operators of position and infinitesimal displacement in space-time. In the x-representation we have

$$<x' \mid x_\mu \mid x'' > = x'_\mu \varrho\,(x' - x''), \qquad <x' \mid d_\mu \mid x'' > = -\frac{\partial}{\partial x'_\mu}\,\delta\,(x' - x'') \tag{14}$$

whence it is easily inferred

$$< x' \mid [x_\mu, F] \mid x'' > = (x'_\mu - x''_\mu) < x' \mid F \mid x'' > \tag{15'}$$

$$< x' \mid [d_\mu, F] \mid x'' > = -\left(\frac{\partial}{\partial x'_\mu} + \frac{\partial}{\partial x''_\mu}\right) < x' \mid F \mid x'' > \tag{15''}$$

for an arbitrary F. With the aid of the above formulae we have

$$\delta_{\varphi^*}({}_1{}_\sigma W) = \int d^4x' d^4x''\, \delta <x' \mid \psi^* \mid x'' > \{(x'_\mu - x''_\mu) <x'' \mid [{}_\sigma u_\mu, \psi] \mid x' > + l^4 M^2 \delta_{1\sigma} <x'' \mid \psi \mid x' >\} \tag{16'}$$

$$\delta_{\varphi^*}({}_2{}_\sigma W) = \int d^4x' d^4x''\, \delta <x' \mid \psi^* \mid x'' > \left\{ l^2 \left(\frac{\partial}{\partial x'_\mu} + \frac{\partial}{\partial x''_\mu}\right) <x'' \mid [{}_\sigma u_\mu, \psi] \mid x' > + l^4 M^2 \partial_{2\sigma} <x'' \mid \psi \mid x' > \right\} \tag{16''}$$

where we have used an obvious generalization of the traditional condition that the variation on the boundary of integration domain vanish. Using again (15) we get from (16)

$$\delta_{\varphi^*}({}_{\varrho\sigma} W) = \int d^4x' d^4x''\, \delta <x' \mid \psi^* \mid x'' > \{- <x'' \mid [{}_\varrho u_\mu, [{}_\sigma u_\mu, \psi]] \mid x' > + l^4 M^2 \delta_{\varrho\sigma} <x'' \mid \psi \mid x' >\}, \tag{17}$$

wherefrom we infer any of the equations (2) written in the x-representation may be.

The fact that the four conditions (11) or (17) are consistent with each other, and that the variations $\delta < x' \mid \psi^* \mid x'' >$ may best be seen in a (p, r) — representation. Let us introduce first the variables

$$x = \frac{1}{2}(x' + x''), \quad r = x' - x''. \tag{18}$$

The representatives of the field quantities may be treated as functions of the variables x, r

$$<x' \mid \psi^* \mid x''> = \psi^*(x, r), \quad <x'' \mid \psi \mid x' > = \psi(x, r) \tag{19}$$

and the action integrals become

$$W_{11} = \int d^4x d^4r \,(-r_\mu r_\mu + l^4M^2)\,\psi^*\psi$$

$$W_{12} = \int d^4x\, d^4r\, \frac{\partial\psi^*}{\partial x_\mu} r_\mu \psi, \qquad W_{21} = -\int d^4x\, d^4r\, r_\mu \psi^* \frac{\partial\psi}{\partial x_\mu} \tag{20}$$

$$W_{22} = l^4 \int d^4x\, d^4r \left(\frac{\partial\psi^*}{\partial x_\mu}\frac{\partial\psi}{\partial x_\mu} + M^2\psi^*\psi\right)$$

where ψ, and ψ^* are functions of x and r. We may apply the variational principle (11) in this representation but it is also possible (and simpler) to go over to the momentum representation. Introducing the Fourier transform

$$\psi(x,r) = \frac{1}{(2\pi)^4}\int d^4p\, e^{ipx}\,\psi(p, r) \tag{21}$$

(20) may be written in the form

$$W_{11} = \int d^4p d^4r\,(-r_\mu r_\mu + l^4M^2)\psi^*\psi$$

$$W_{12} = W_{21} = -i\int d^4p d^4r p_\mu r_\mu \psi^*\psi \tag{20'}$$

$$W_{22} = l^4\int d^4p d^4r\,(p_\mu p_\mu + M^2)\,\psi^*\psi$$

Now ψ and ψ^* are functions of p_μ and r_μ. This last form is particularly simple since no integrations by parts need be performed and the Euler-Lagrange equations are algebraic equations

$$(-r_\mu r_\mu + l^4M^2)\,\psi = p_\mu r_\mu \psi = (p_\mu p_\mu + M^2)\,\psi = 0 \tag{22}$$

obviously compatible with each other. The solution of (22) is

$$\psi(p, r) = \delta(p_\mu^2 + M^2)\,\delta(p_\mu r_\mu)\,\delta(-r_u^2 + l^4M^2)\,\varphi(p, r) \tag{23}$$

where $\varphi(p, r)$ is an arbitrary function. In consequence of the appearance of three δ-functions in (23), only the dependence of φ on five of the eight variables p, r is essential. These five variables ($\vec{p}$ and two of the $r'_\mu s$) may be called „independent".

The above considerations may easily be generalized for the case where M is not a c-number but an operator $\boldsymbol{M}$. In a representation $\boldsymbol{M}$ will be a differential operator. The compatibility with the field equations will be ensured if the mass operator involves only derivatives with respect to the „independent" variables. In particular, in the (p, r)-representation, the mass operator must commute with the arguments of the three δ-functions appearing in (23), i.e. it must involve only derivatives with respect to two of the four $r'_\mu s$, namely, the two „independent ones. In this case the mass eigen-equation

$$\boldsymbol{M}\,\psi(p, r) = M\psi(p, r) \tag{24}$$

may be written as a condition upon the function $\varphi(p, r)$ appearing on the right hand side of (23)

$$\boldsymbol{M}\,\varphi(p, r) = M\,\varphi(p, r) \tag{24'}$$

In the traditional field theory, it was possible to derive from the action integral not only the field equations, but also the conservation laws. Corresponding methods for deriving conservation laws in the present case have not yet been worked out. Nevertheless, the conservation laws, in the usual form of continuity equations, follow directly from the field equations. The conserved quantities assume a particularly simple form in the (p, r)-representation, as was shown in a previous paper (Rayski 1955).

КРАТКОЕ СОДЕРЖАНИЕ

Г. Райский. *Вариационный принцип в билокальной теории поля.*

Обобщённые уравнения Schrödingera-Gordona-Focka билокальной теории поля получены, как уравнения Eulera-Lagrange'a, надлежаще подобранного вариационного принципа.

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A SIMPLE NUCLEAR MODEL FOR ^{7}Li AND ITS USE IN INVESTIGATING THE REACTION ^{7}Li (γ, ^{3}H) ^{4}He

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For computing the total and differential cross sections for the reaction ^{7}Li (γ, ^{3}H)^{4}He, a 'two-body' nuclear model was used fot ^{7}Li. It has been assumed for this model that the ^{7}Li nucleus can be regarded as a compound system composed of a triton and an α particle rotating around the common centre of mass. The cross section curve so obtained basically agrees with the results obtained experimentally if we assume that the forces due to the interaction of the triton with the α particle are repulsive in the D state.

I. Introduction

The reaction ^{7}Li (γ,^{3}H)^{4}He was discovered in 1950 by Titterton (Titterton 1950), and subsequently investigated experimentally by many authors (Nabholz et al. 1952, Stoll et al. 1953, Titterton et al. 1953, Stoll 1954, Erdös et al. 1954). No theoretical account for this reaction has so far appeared. The experimental data are as yet incomplete: for incident γ-rays of energies $\hbar\omega \lesssim 4$ MeV, the cross section is not known; for energies $\hbar\omega \gtrsim 11$ MeV, only two cross sections are known (for $\hbar\omega =$ $= 14.8$ and 17.6 MeV); the angular distributions still haven't been accurately determined (Stoll 1954, Erdös et al. 1954). Nevertheless, several characteristic features of this reaction can be noted (see the points marked on Fig. 1). The cross section has a broad minimum for incident γ-rays having energies in the neighbourhood of $\hbar\omega \approx$ ≈ 11 MeV. To the left of this region, a maximum of $2 \cdot 10^{-28}$ cm^2 is reached. The existance of a series of resonances (Stoll 1954, Erdös et al. 1954) (not shown in Fig. 1) has been found in the energy range $0 < \hbar\omega \lesssim 11$ MeV.

The angular distribution for 4.7 MeV γ-rays closely follows the empirical function $1-0.5\cos^2\theta$ (θ is the angle between the direction of the incident radiation and the direction of the ejected triton). On the other hand, the same function is derived theoretically as an angular distribution for certain 'definite'[1] multipole transitions,

[1] By this term, we mean the transition from a state with one definite total and orbital angular momentum to a second state with another definite total and orbital angular momentum.

(. e., the electric dipole transition ($P_{3/2} \rightarrow D_{3/2}$) and magnetic dipole transition $iP_{3/2} \rightarrow G_{3/2}$). The empirically obtained angular distribution for other energies follows a more complicated relation and, as a result of considerable errors in measurement, could be represented by an entire series of 'definite' multipole transitions (see Stoll 1954).

In this paper, the cross section for the reaction ^{7}Li (γ, t) α has been calculated, and the angular distribution determined, by the use of a two-body model for the ^{7}Li nucleus. In this model, the ^{7}Li nucleus is regarded as a compound system composed of a triton and an α particle rotating about the common centre of mass. This model was proposed by Bethe (Bethe 1938) and Inglis (Inglis 1941) a long time ago, and it can be used to explain the spin and magnetic moment of the ^{7}Li nucleus. The most important arguments supporting the $t - \alpha$ model for the ^{7}Li nucleus are the following:

1. The ^{7}Li nucleus separates most easily into a triton and an α particle. The reaction threshold for ^{7}Li (γ, t) α (2.465 MeV) is considerably lower than the thresholds for the reactions (γ, n) and (γ, p) which are 7.15 and 9.5 MeV respectively.

2. With the help of the $t - \alpha$ model, it is easy to arrive at the ground state $P_{3/2}$ of the ^{7}Li nucleus: the triton and the α particle are in the P state; we obtain the value of $^3/_2$ for the spin of the ^{7}Li nucleus by adding the orbital angular momentum (equal to 1) to the spin of the triton (equal to $^1/_2$) (α particles have a spin of 0).

3. A system formed by a triton (magnetic moment 2.9789) and an α particle (magnetic moment 0) should have, in state $P_{3/2}$ (orbital magnetic moment 0.405), a magnetic moment of 3.3839 nuclear magnetons, which is equal to the sum of the three previously mentioned magnetic moments; the magnetic moment found experimentally for the ^{7}Li nucleus is equal to 3.2559 nuclear magnetons, a difference of only 3% from the value calculated on the basis of the $t - \alpha$ model.

The $t - \alpha$ model for the ^{7}Li nucleus is a very rough model and too much cannot be expected from it. Nevertheless, the results of this investigation seem to indicate that the 'two-body mechanism' for the reaction ^{7}Li (γ, t) α, according to which the incident photon knocks apart the triton and α particle making up the ^{7}Li nucleus, provides a good explanation for the order of magnitude of the cross section and the basic shape of the cross section curve: — two maxima separated by a broad minimum. It is very doubtful if the already mentioned series of resonances observed by Stoll can be obtained from this model. On the basis of the considerations made below, the conclusion can be drawn that these resonances may be, in principle, the result of magnetic dipole, electric quadrupole, and higher order multipole transitions. This question might be settled only after carrying out long and tedious numerical computations. The calculating machines which the author had at his disposal did not allow the carrying out of such computations. It is, however, very probable that the reaction ^{7}Li (γ, t) α can have a second, competing mechanism in addition to the 'two-body mechanism' described above, e. g., the reaction may involve a compound nucleus. Then, the observed, fine structure' of the cross section can be ascribed to

the second mechanism. The cross section would then be a superposition of cross sections stemming from the first and second mechanisms. The coexistence of these two types of mechanisms is already known for various other photo-nuclear reactions (see Courant 1951).

The $t - \alpha$ system has an electic dipole moment (in the same sense as the deuteron, for example). The predominent transitions will, therefore, be electric dipole transitions. It should be noted that tritons lack only one proton for saturated nuclear forces to occur, and therefore, the proton tends to tear away from the α particle, the latter now becoming a triton which in turn, gives rise to a tendency towards saturation. In other words, it appears that neither of the two components entering into the nuclear structure of ^{7}Li will be distinguishable, and the mean electric dipole moment will vanish. Nevertheless, the $t - \alpha$ model can be used for calculating the cross sections for photons whose periods of vibration are shorter than the half- life (which we denote by T) of the α particle in the ^{7}Li nucleus. This time has been estimated by Wheeler (Wheeler 1937) to be about 10^{-21} sec. (this is actually the lower limit of his estimation). This period corresponds to the γ-ray energy

$$h\nu_{crit} = \hbar\omega_{crit} \approx 0.66 \times 10^{-21} \times \text{MeV sec} \times 2\pi \times 10^{21} \times \text{sec}^{-1} \approx 4\text{MeV}$$

As a result, it appears that the $t - \alpha$ model can be used to study reactions with photons of energy greater than 4 MeV, while the applicability of this model for low energy photons is doubtful.

In the calculations, the Coulomb interaction energy for the triton and α particle was neglected. (This greatly simplifies the analysis.) The results, therefore, serve rather for orientational purposes. The mathematical treatment is based partly on the paper by Guth and Mullin (Guth 1949). The symbols introduced in this paper are, as far as possible, the same as those used by Guth and Mullin (Guth 1949). Calculation of the Coulomb correction will be the topic of an other paper.

II. Differential and total cross section for the reaction $^7Li\,(\gamma, t)\,\alpha$

We assume that the interaction $V(r)$ between the triton and the α particle has the shape of a rectangular potential well:

$$V(r) = -V \qquad r < r_0$$

$$V(r) = \quad 0 \qquad r > r_0$$

In addition, we assume that the interaction between the triton and the α particle is a function of the orbital angular momentum of the $t - \alpha$ system.

For the states of various angular momentum, we will choose different depths of the potential well and use subscripts to distinguish between the various values of V. Thus V_S, V_P, V_D, etc., represent the depth of the potential well for the states S, P, D, etc.

In our treatment, the radius of the well r_0 will be taken to be independent of the

momentum of the system[1]. The assumption that the interaction energy is a function of the momentum of the system is quite essential. Without it, the agreement between theory and experiment will be upset.

The wave functions with given quantum numbers j, l, m can be written with the help of the Clebsch — Gordan coefficients:

$$\psi\left(l, j = l \pm \frac{1}{2}, m_j; r, \vartheta, \varphi\right) = R_{lj}(r) \sum_{\sigma_z} C_{l,1/2}(j, m_j; m_j - \sigma_z, \sigma_z) \times$$

$$\times Y_{l, m_j - \sigma_z}(\vartheta, \varphi)\, \chi(\sigma_z) = \frac{R_{lj}(r)}{(2l+1)^{1/2}} \left\{ \left(l + \frac{1}{2} \pm m_j\right)^{1/2} Y_{l, m_j - 1/2}(\vartheta, \varphi)\, \alpha \pm \right.$$

$$\left. \pm \left(l + \frac{1}{2} \mp m_j\right)^{1/2} Y_{l, m_j + 1/2}(\vartheta, \varphi)\, \beta \right\} \tag{1}$$

where R_{lj} is the radial part of the wave function; Y_{lm} are the normalized surface harmonics; $\chi(\sigma_z)$ is the spin function, for wich is used the generally accepted notation

$$\chi\left(\sigma_z = \frac{1}{2}\right) = \alpha; \quad \chi\left(\sigma_z = -\frac{1}{2}\right) = \beta$$

Since the interaction was chosen in the form of a rectangular potential well, it should be possible to obtain the radial parts of the wave function R_{lj} explicitly from the Schrödinger equation. In our case, for example, the ground state is $P_{3/2}$; for $R_{1,3/2}$ we obtain the expression

$$R_{1,3/2}(r) = A_1 j_1(\beta r) \qquad r < r_0$$

$$R_{1,3/2}(r) = B_1 \frac{(1 + \alpha r)}{r} e^{-\alpha(r - r_0)} \qquad r > r_0 \tag{2}$$

where $\alpha = \sqrt{\frac{2\mu}{\hbar^2}\varepsilon}$, $\beta = \sqrt{\frac{2\mu}{\hbar^2}(V_p - \varepsilon)}$, j_1 spherical Bessel function, ε is the binding energy (in our case, equal to 2.465 MeV) and μ is the reduced mass of the triton — α particle system. In addition, we also have the following relations resulting from the condition that the functions join smoothly at the boundary of the well:

$$B_1 = -\frac{A_1}{\alpha^2} \sin \beta r_0$$

$$\beta r_0 \operatorname{ctg} \beta r_0 = 1 + (1 + \alpha r_0)\left(\frac{\beta}{\alpha}\right)^2$$

$$\frac{A_1^2 r_0}{2\beta^2}\left\{1 + \left[(2 + \alpha r_0)\left(\frac{\beta}{\alpha}\right)^4 + (1 + \alpha r_0)\left(\frac{\beta}{\alpha}\right)^2 - 1\right] \frac{\sin^2 \beta r_0}{(\beta r_0)^2}\right\} = 1 \tag{3}$$

[1] This assumption is analogous to that made by Guth and Mullin (Guth 1949) for the ^{8}Be — n interaction. The difference between these assumptions lies only in the fact that in our case, we have assumed that V depends on the momentum and not only on the parity as in the case of Guth and Mullin. It is not excluded, however, that the Coulomb correction may change this situation to the extent that in our case, V will depend only on the parity (cf. conclusion of this paper).

We will now evaluate the differential cross section for our reaction. The differential cross section is

$$d\sigma = v < | \lim_{r \to \infty} \Psi(\vec{r}, t) |^2 > r^2 \, d\Omega \tag{4}$$

where v is the relative velocity of the triton and the α particle after the reaction, $\Psi(\vec{r}, t)$ is the wave function of the ground state of the ^{7}Li nucleus disturbed by the electromagnetic field of the incident photon. $< \; >$ indicates that we take the mean of the initial states and average over all directions of polarization of the electromagnetic waves. $\Psi(\vec{r}, t)$ is normalized to unity. We take the electromagnetic wave disturbing the system to be a plane wave with an amplitude corresponding to an incident photon stream of unit intensity.

The predominating transitions are electric dipole transitions (see Guth 1949). As we know from experiment, the ground state of the ^{7}Li nucleus is $P_{3/2}$. Therefore, electric dipole transitions can lead to states $D_{5/2}$, $D_{3/2}^1$ and $S_{1/2}$. As a result, we will consider, in addition to the function representing the ground state, only those terms of $\Psi(\vec{r}, t)$ containing the functions $\psi(2, {}^5/_2, m)$, $\psi(2, {}^3/_2, m)$, $\Phi(0, {}^1/_2, m)$ belonging to the continuous spectrum. Since we are interested only in the asymptotic form of $\Psi(\vec{r}, t)$ at large values of r (see Bethe 1930) we will be dealing, finally, with the superposition

$$\Psi(\vec{r}, t) \approx \int \left[a_m(E') \, \psi_{E'} \left(2, \frac{5}{2}, m\right) + b_m(E') \, \psi_{E'} \left(2, \frac{3}{2}, m\right) + C_m(E') \, \psi_{E'} \left(0, \frac{1}{2}, m\right) \right] dE' \tag{5}$$

where E' represents the kinetic energy of the triton — α particle system. The wave function of the bound state vanishes more rapidly than r^{-1} and will not play any important role in Eq. (4). We will normalize the function ψ_E belonging to the continuous spectrum to $\delta(E - E')$. This means

$$\int \psi_E^* \, \psi_{E'} \, d\tau = \delta(E - E') \tag{6}$$

We determine the coefficients $a_m(E)$, $b_m(E)$, $C_m(E)$ by the wellknown method used in Dirac's radiation theory. The summation over the energy will be performed according to Bethe (Bethe 1930). As is known from radiation theory, these coefficients are proportional to the dipole moment matrix elements of the triton — α particle system. This leads to the integral

$$z(l, j, m \mid l', j', m) = \int \psi^*(l', j', m) \, z \, \psi(l, j, m) \, d\tau \tag{7}$$

$\psi(l, j, m)$ are given by Eq. (1), where R_{lj} may (in the case of a rectangular potential

[1] In the calculations, we don't consider the possible splitting of the $D_{5/2}$ and $D_{3/2}$ states.

well) be given analytically. We will use the same form employed by Guth and Mullin (Guth 1949)

$$R_{0,1/2} = A_0 j_0(\gamma r) \qquad r < r_0$$

$$R_{0,1/2} = \left(\frac{2\mu k}{\pi\hbar^2}\right)^{1/2} \frac{\sin\left[k(r-r_0)+\delta_0\right]}{kr} \qquad r > r_0 \tag{8}$$

where $k = \left(\frac{2\mu}{\hbar^2} E\right)^{1/2}$, $\gamma = \left[\frac{2\mu}{\hbar^2}(V_S + E)\right]^{1/2}$

$$A_0 \sin\gamma r_0 = \frac{\gamma}{k}\left(\frac{2\mu k}{\pi\hbar^2}\right)^{1/2} \sin\delta_0$$

$$A_0 \cos\gamma r_0 = \left(\frac{2\mu k}{\pi\hbar^2}\right)^{1/2} \cos\delta_0 \tag{9}$$

$$R_{2,3/2} = R_{2,5/2} = A_2 j_2(\xi r) \qquad r < r$$

$$R_{2,3/2} = R_{2,5/2} = \left(\frac{2\mu k}{\pi\hbar^2}\right)^{1/2} \frac{(3-k^2r^2)\sin\left[k(r-r_0)+\delta_2\right] - 3kr\cos\left[k(r-r_0)+\delta_2\right]}{k^3 r^3}$$

$$r > r_0 \tag{10}$$

and where $\xi = \left[\frac{2\mu}{\hbar^2}(V_D + E)\right]^{1/2}$

$$A_2 = \left(\frac{2\mu k}{\pi\hbar^2}\right)^{1/2}\left(\frac{\xi}{k}\right)^3 \frac{(3-k^2r_0^2)\sin\delta_2 - 3kr_0\cos\delta_2}{(3-\xi^2r_0^2)\sin\xi r_0 - 3\xi r_0\cos\xi r_0}$$

$$\frac{(3-k^2r_0^2)\sin\delta_2 - 3kr_0\cos\delta_2}{(3-\xi^2r_0^2)\sin\xi r_0 - 3\xi r_0\cos\xi r_0} = \left(\frac{k}{\xi}\right)^2 \frac{\sin\delta_2 - kr_0\cos\delta_2}{\sin\xi r_0 - \xi r_0\cos\xi r_0} \tag{11}$$

The j_0, j_2 in Eqs. (8) to (11) are spherical Bessel functions. $R_{2,3/2} = R_{2,5/2}$ since we have assumed that the interaction of the triton and α particle is a function of the angular momentum of the system only. (We neglect the spin-orbit interaction).

We are now in a position to find the matrix element $z(l, j, m \mid l, j, \mathrm{m})$. Thus after elementary, but cumbersome integration, we obtain

$$z\left(0, \frac{1}{2}, \frac{1}{2}\middle| 1, \frac{3}{2}, \frac{1}{2}\right) = \frac{\sqrt{2}}{3}\int_0^\infty r^3 R^*_{0,1/2} R_{1,3/2}\, dr = \frac{\sqrt{2}}{3} R_{PS},$$

where

$$R_{PS} = \frac{B_1\left[\frac{2\mu k}{\pi\hbar^2}\right]^{1/2}}{r_0\left(k^2+\gamma^2\operatorname{ctg}^2\gamma r_0\right)^{1/2}}\left\{\frac{2\alpha^2 + 2\beta^2(1+\alpha r_0) - \alpha^2 r_0^2(\gamma^2-\beta^2)}{(\gamma^2-\beta^2)^2} + \right.$$

$$+ \frac{\alpha r_0(2+\alpha r_0)\alpha^2 + k^2 r_0^2\alpha^2}{(\alpha^2+k^2)^2} + \left[-\frac{2\alpha^2 + (\gamma^2-\beta^2)(1+\alpha r_0)}{(\gamma^2-\beta^2)^2} + \right.$$

$$\left.\left. + \frac{(3+\alpha r_0)\alpha^2 + k^2(1+\alpha r_0)}{(\alpha^2+k^2)^2}\right]\gamma r_0 \operatorname{ctg}\gamma r_0\right\}, \tag{12}$$

B_1 is given by Eq. (3). Similarly,

$$z\left(2,\frac{5}{2},\frac{3}{2}\,\middle|\,1,\frac{3}{2},\frac{3}{2}\right)=\frac{2}{5}\int_0^\infty r^3\,R^*_{2,5/2}\,R_{1,5/2}\,dr=\frac{2}{5}\,R_{PD}\,,$$

where

$$R_{PD}=\frac{B_1\left[\dfrac{2\mu k}{\pi\hbar^2}\right]^{1/2}}{\{k^2r_0^2\,(3QP+k^2r_0^2)^2+(3QP+k^2r_0^2\,\xi r_0\,\mathrm{ctg}\,\xi r_0)^2\}^{1/2}}\times$$

$$\times\left\{k^2r_0^2\left[\frac{2k^2\,(1+\alpha r_0)-\alpha^2r_0^2\,(\alpha^2+k^2)}{(\alpha^2+k^2)^2}-\frac{2\,(1+\alpha r_0)\,\xi^2-\alpha^2r_0^2\,(\xi^2-\beta^2)}{(\xi^2-\beta^2)^2}\right]+\right.$$

$$+k^2r_0^2\,Q\left[\frac{(\alpha^2+k^2)\left(1+\alpha r_0+3\dfrac{\alpha^2}{k^2}\right)+2\alpha^2}{(\alpha^2+k^2)^2}-\right.$$

$$\left.-\frac{(\xi^2-\beta^2)\left(1+\alpha r_0+\dfrac{\alpha^2}{\xi^2}\right)+2\alpha^2}{(\xi^2-\beta^2)^2}\right]+$$

$$\left.+3PQ\left[\frac{2k^2\,(1+\alpha r_0)-\alpha^2r_0^2\,(\alpha^2+k^2)}{(\alpha^2+k^2)^2}\right]\right\}\qquad(13)$$

and where $P=1-\dfrac{\xi^2}{k^2}$, $Q=1-\xi r_0\,\mathrm{ctg}\,\xi r_0$.

For the remaining $z\,(l,j,m\mid l',j'\,m)$, we obtain the expressions

$$z\left(0,\frac{1}{2},-\frac{1}{2}\,\middle|\,1,\frac{3}{2},-\frac{1}{2}\right)=z\left(0,\frac{1}{2},\frac{1}{2}\,\middle|\,1,\frac{3}{2},\frac{1}{2}\right)=\frac{\sqrt{2}}{3}\,R_{PS}$$

$$z\left(2,\frac{5}{2},-\frac{3}{2}\,\middle|\,1,\frac{3}{2},-\frac{3}{2}\right)=z\left(2,\frac{5}{2},\frac{3}{2}\,\middle|\,1,\frac{3}{2},\frac{3}{2}\right)=\frac{2}{5}\,R_{PD}$$

$$z\left(2,\frac{5}{2},\frac{1}{2}\,\middle|\,1,\frac{3}{2},\frac{1}{2}\right)=z\left(2,\frac{5}{2},-\frac{1}{2}\,\middle|\,1,\frac{3}{2},-\frac{1}{2}\right)=\frac{3}{5}\sqrt{\frac{2}{3}}\,R_{PD}\qquad(13a)$$

$$z\left(2,\frac{3}{2},\frac{3}{2}\,\middle|\,1,\frac{3}{2},\frac{3}{2}\right)=-\,z\left(2,\frac{3}{2},-\frac{3}{2}\,\middle|\,1,\frac{3}{2},-\frac{3}{2}\right)=\frac{1}{5}\,R_{PD}$$

$$z\left(2,\frac{3}{2},\frac{1}{2}\,\middle|\,1,\frac{3}{2},\frac{1}{2}\right)=-\,z\left(2,\frac{3}{2},-\frac{1}{2}\,\middle|\,1,\frac{3}{2},-\frac{1}{2}\right)=\frac{1}{15}\,R_{PD}$$

Using Eqs. (4), (5), (12), and (13), we obtain (see Appendix *A*) the following expression for the differential cross section:

$$\sigma(\theta)=\frac{\sigma_{PD}}{100\pi}\,(17+12\sin^2\theta)+\frac{1}{4\pi}\,\sigma_{PS}+\frac{\sigma_{PD}}{100\pi}\,\frac{9}{2}\left(\frac{3}{2}\sin^2\theta-1\right)\mp$$

$$\mp\frac{1}{4\pi}\,\sqrt{2\sigma_{PS}\,\sigma_{PD}}\cos(\delta_2-\delta_0)\left(\frac{3}{2}\sin^2\theta-1\right)\qquad(14)$$

where θ is the angle between the direction of the incident photon and the direction of the ejected triton. σ_{PS} and σ_{PD} are the total cross sections for the transitions $P_{3/2} \to D_{3/2}$, $D_{5/2}$ and $P_{3/2} \to S_{1/2}$, and are connected with R_{PD} and R_{PS} (see Appendix B) by the relations

$$\sigma_{PD} = \pi^2 \frac{e^2}{\hbar c} \hbar\omega \frac{16}{49} \times \frac{10}{45} R_{PD}^2$$
$$\sigma_{PS} = \pi^2 \frac{e^2}{\hbar c} \hbar\omega \frac{16}{49} \times \frac{1}{9} R_{PS}^2 \tag{15}$$

Integrating Eq. (14) over θ we get the total electric dipole cross section

$$\sigma = \sigma_{PD} + \sigma_{PS} \tag{16}$$

It can easily be shown that the third and fourth terms in Eq. (14) do not make any contribution to σ. These terms[1] are responsible for interference between states $D_{3/2}$, $D_{5/2}$, and $S_{1/2}$.

Numerical computations were made for a well radius of $r_0 = 5 \times 10^{-13}$ cm. Since the binding energy of the $P_{3/2}$ state is 2.465 MeV, the depth of the well for this state can now be determined single-valuedly from Eq. (3). We thus arrive at the value $V_p = -8.45$ MeV. As already mentioned, in order to obtain agreement with experimental data, it is necessary to assume that the well depth is a function of the angular momentum of the triton — α particle system.

In Figs. 1 and 2, σ_{PS} and σ_{PD} are drawn as functions of the incident γ-ray energy for different well depths: $V_S = -1$ MeV, and $+4$ MeV, and $V_D = -2$ MeV, 0 MeV, and $+4$ MeV.

Figs. 1 and 2 show that in order to obtain good agreement with experimental results, it should be assumed that repulsive forces exist between the triton and α particle in the D state ($V = +4$ MeV). On the other hand, it appears that for the S state of the $t - \alpha$ system, attractive forces should be assumed ($V = -1$ MeV). This can be shown by the following argument. In Ref. (Stoll 1954) it was found that the triton angular distribution for 4.7 MeV incident γ-rays is closely represented by the function $1 - 0.5 \cos^2\theta$. If we had assumed that the magnitude of the $t - \alpha$ interaction depends only on the parity of the state as Guth and Mullin (Guth1949) had done for the ^{8}Be — n interaction, i. e., $V_S = V_D = +4$ MeV, then for 4.7 MeV γ-rays, we would have $\sigma_{PS} \approx \sigma_{PD}$ (see Fig. 1). Eq. (14) gives us the angular distribution. δ_2 and δ_0 are determined by the required smoothness of the function at the boundary of the well. We normalize the result and obtain for the angular distribution,

[1] The solution for the differential cross section is found here in a manner completely different from that given in the paper of Guth and Mullin (Guth 1949). The result given by Eq. (14) differs from the analogous equation in Guth and Mullin's paper precisely by the interference terms. In light of the derivations given above and in Appendix A, it appears that the differential cross section for the reaction ^{9}Be (γ, n) ^{8}Be given Ref. (Guth 1949) is not correctly calculated.

the function $1 + 0.369 \cos^2 \theta$. It is, therefore, very probable that for 4.7 MeV γ-rays, we are dealing only with the transition $P \rightarrow D$. If so, then it is necessary to choose the potential for state S so as to decrease considerably σ_{PS} for an energy of 4.7 MeV. This can be done by taking negative values for V_S. (The deeper the well, the closer σ_{PS} approaches the threshold of the reaction). We chose $V_S = -1$ MeV since a bound

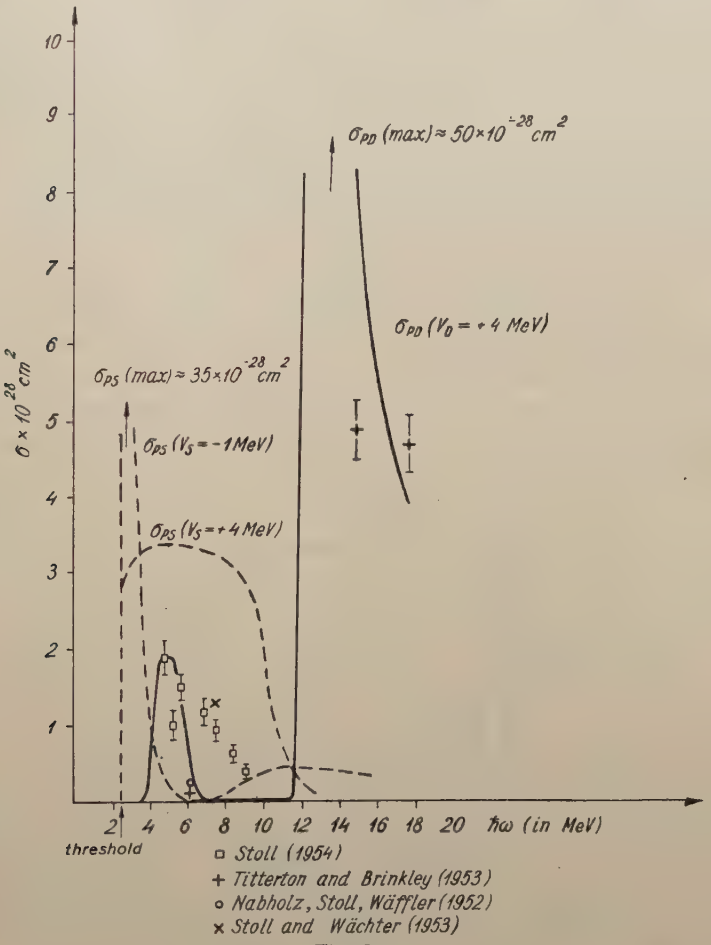

Fig. 1.

state arises for the $t - \alpha$ system when $V_S = -1.2$ MeV and the ^{7}Li nucleus is not known to have a bound S state. Therefore, by finally choosing $V_P = -8.45$ MeV, $V_S = -1$ MeV, $V_D = 4$ MeV, we obtain the curve of Fig. 1. For energies $\hbar\omega \gtrsim 4$ MeV, the cross section curve for the reaction $^7\mathrm{Li}\,(\gamma, t)\,\alpha$ is, therefore, essentially that of σ_{PD} and the angular distribution is represented by the function $1 - 0.6 \cos^2 \theta$.

It should, however, be made clear that in arriving at the above conclusions, the Coulomb interaction between the triton and the α particle was neglected. The qualitative influence of the Coulomb interaction can be described as follows:

1. The largest correction to the curves in Fig. 1 will occur for low energies.
2. The effect of the Coulomb interaction will be to lower the values of σ_{PS} and σ_{PD}.
3. The Coulomb interaction will have a considerably greater influence on σ_{PS} than on σ_{PD}.

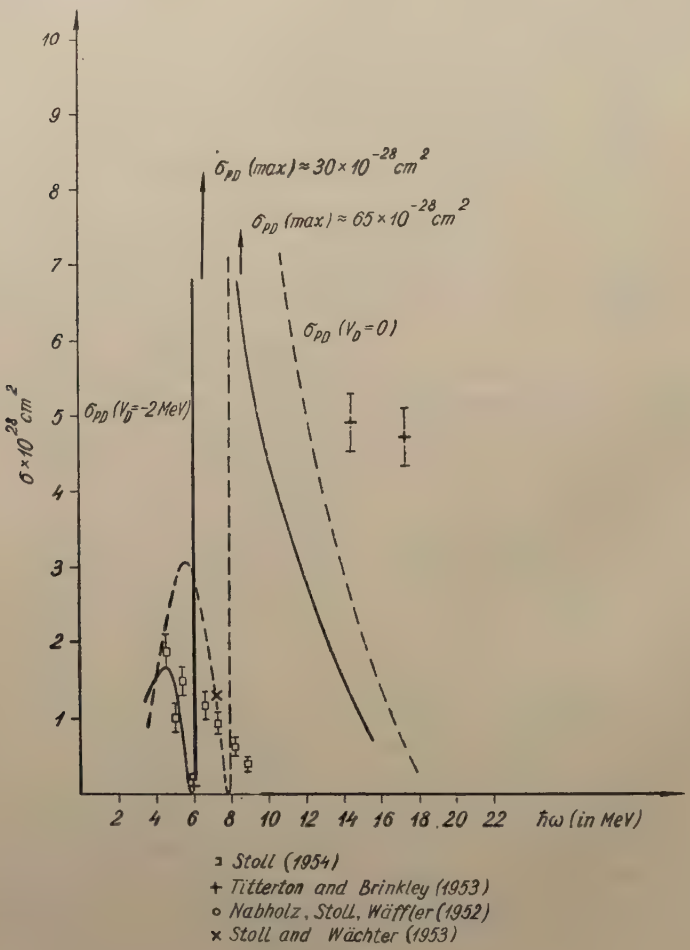

Fig. 2.

Such assumptions can be made on the basis of the known properties of the Coulomb interaction for other nuclear reactions. It is, therefore, possible that the values of σ_{PS} decrease in such a way that the triton — α particle nuclear interaction can remain a function of the parity of the state alone, and we can be set $V_S = V_D = +4$MeV. In spite of this, the shape of the cross section *vs* energy curve will still be determined by the transition $P \rightarrow D$.

It is necessary to discuss still one more side of the problem. The existence of a resonance has been established empirically for 4.7 MeV γ-rays (see Stoll 1954),

and the angular distribution $1-0.5\cos^2\theta$ is the distribution for the resonant photodisintegration of ^{7}Li. At the beginning of this paper, we put forward the hypothesis that there are two mechanisms for the reaction; one of these is the 'two-body mechanism'; the other is the compound nucleus mechanism which gives a sharp resonance on the background of the slowly varying cross section *vs* energy curve associated with the 'two-body mechanism'. If this is the case, then the distribution $1 - 0.5\cos^2\theta$ is a superposition (non-coherent, since we are dealing with two different mechanisms) of the distributions arising from both mechanisms. The distribution $1 - 0.5\cos^2\theta$ can be obtained, as already mentioned in the introduction, from two 'definite' multipole transitions: the electric dipole transition $P_{3/2} \rightarrow D_{5/2}$ and the magnetic dipole transition[1] $P_{3/2} \rightarrow G_{5/2}$. Thus, we obtain good agreement with experimental results if we assume that the resonance for an energy of 4.7 MeV is associated with the states $D_{5/2}$ or $G_{5/2}$ of the compound nucleus; the angular distributions associated with these transitions when added to the angular distribution associated with the 'two-body mechanism' gives a distribution very close to $1 - 0.5\cos^2\theta$.

Finally, we estimate the magnitude of the cross section for the inverse reaction, namely, the capture of a triton by an α particle, accompanied by the emission of a γ quanta. If we denote the cross section for this reverse process by σ_c, then σ_c will be related to the cross section σ for photodisintegration by the formula (see Blatt 1952):

$$\sigma_c = \left(\frac{\varkappa}{k}\right)^2 \sigma \tag{17}$$

where $\varkappa = \dfrac{\omega}{c}$ and $k = \sqrt{\dfrac{2\mu}{\hbar^2} E}$

The factor $\left(\dfrac{\varkappa}{k}\right)^2$ can be expressed in a different manner:

$$\left(\frac{\varkappa}{k}\right)^2 = \frac{(\varepsilon + E)^2}{2\mu c^2 E} \tag{18}$$

In our case, $2\mu c^2 \approx 3000$ MeV. Factor (18) is large for very small energies ($E \ll 1$ MeV). But for these energies, the Coulomb interaction between the triton and the α particle reduces σ practically to zero, while for energies of the order of several MeV, the factor (18) is of the order 0.01. For example, for $E = 5.5$ MeV,

$$\frac{(\varepsilon + E)^2}{2\mu c^2 E} \approx \frac{64}{3000 \times 5.5} \approx \frac{1}{300}$$

Since within the range of energies considered here, σ is of the order 10^{-28} cm^2, the

[1] The angular distributions for the 'definite' multipole transitions do not depend on the mechanism of the reaction, but only on the angular momentum and parity of the initial and final states of the nucleus. These angular distributions can be obtained from the general correlation formulae (see Stoll 1954).

cross section for the capture of tritons by α particles can finally be estimated as

$$\sigma_c \approx 10^{-30} \text{ cm}^2$$

Consequently, the probability of this inverse reaction is very small.

I would like to thank Professor J. Weyssenhoff for discussion of the problems considered in this paper.

Appendix A

We calculate differential cross section on the basis of the results obtained by Bethe (see Bethe 1930, and Bethe 1933).

$$\sigma(\theta) \sim < \left| \int_{r\to\infty} \left[z\left(2, \frac{5}{2}, m \mid 1, \frac{3}{2}, m\right) \psi_{E'}\left(2, \frac{5}{2}, m\right) + \right.\right.$$

$$\left.\left. + z\left(2, \frac{3}{2}, m \mid 1, \frac{3}{2}, m\right) \psi_{E'}\left(2, \frac{3}{2}, m\right) + z\left(0, \frac{1}{2}, m \mid 1, \frac{3}{2}, m\right) \psi_{E'}\left(0, \frac{1}{2}, m\right) \right] \times \right.$$

$$\left. \times \, dE' \right|^2 > \qquad \text{(A1)}$$

Integration over E'eliminates from the wave function $\psi_{E'}$ the part corresponding to an impinging wave, and leaves only the part corresponding to the outgoing wave (see Bethe 1930). Therefore,

$$\sigma(\theta) \sim < \left| z\left(2, \frac{5}{2}, m \mid 1, \frac{3}{2}, m\right) \left\{\psi_E\left(2, \frac{5}{2}, m\right)\right\}_{out} + z\left(2, \frac{3}{2}, m \mid 1, \frac{3}{2}, m\right) \times \right.$$

$$\left. \times \left\{\psi_E\left(2, \frac{3}{2}, m\right)\right\}_{out} + z\left(0, \frac{1}{2}, m \mid 1, \frac{3}{2}, m\right) \left\{\psi_E\left(0, \frac{1}{2}, m\right)\right\}_{out} \right|^2 > \qquad \text{(A2)}$$

where $\{\psi_E\}_{out}$ represents that part of the asymptotic form for the wave function ψ_E corresponding to the outgoing wave. In determining $\{\psi_E\}_{out}$, we are interested only in the radial part of ψ_E. For the three states appearing in our discussion ($D_{3/2}$, $D_{5/2}$, $S_{1/2}$), we have (The function R_2 represents the radial part of states $D_{3/2}$ and $D_{5/2}$):

$$\{R_0\}_{out} \sim \left(\frac{2\mu k}{\pi\hbar^2}\right)^{1/2} \frac{1}{k} \frac{e^{ikr}}{2ir} e^{i(\delta_0 - kr_0)}$$

$$\{R_2\}_{out} \sim -\left(\frac{2\mu k}{\pi\hbar^2}\right)^{1/2} \frac{1}{k} \frac{e^{ikr}}{2ir} e^{i(\delta_2 - kr_0)} \qquad \text{(A3)}$$

At the present time, we are interested in obtaining the proper angular distribution in a form that doesn't include any of the common factors appearing in the terms of the sum (A. 1). It is then easy to pass over to the differential cross section by multiplying the angular distribution thus obtained by a suitable factor.

We insert (A3) into (A2), take the summation over the magnetic quantum number m and take the average over all directions of polarization. The $z(l, j, m; l', j', m)$ were given by Eqs. (12) and (13), and the angular parts of the function ψ_E

are given by Eq. (1). In this manner, after elementary but involved mathematical manipulation, we obtain

$$\sigma(\theta) \sim R_{PD}^2 \frac{2}{9 \cdot 25}(17 + 12 \sin^2\theta) + \frac{1}{9} R_{PS}^2 + \\ + R_{PD}^2 \frac{1}{25}\left(\frac{3}{2}\sin^2\theta - 1\right) - \\ - R_{PD}\, R_{PS} \frac{2}{9} \cos(\delta_2 - \delta_0)\left(\frac{3}{2}\sin^2\theta - 1\right) \tag{A4}$$

We multiply the right-hand side by $\frac{\sigma_{PD}}{4\pi}\frac{9}{2R_{PD}^2}$ (in order that the right-hand side of Eq. (A4) lead to $\sigma_{PS} + \sigma_{PD}$ after integration over the angle θ) and use the relation (see Appendix B)

$$\frac{\sigma_{PD}}{\sigma_{PS}} = 2\frac{R_{PD}^2}{R_{PS}^2} \tag{A5}$$

We finally obtain[1]

$$\sigma(\theta) = \frac{\sigma_{PD}}{100\pi}(17 + 12\sin^2\theta) + \frac{1}{4\pi}\sigma_{PS} + \\ + \frac{\sigma_{PD}}{100\pi}\frac{9}{2}\left(\frac{3}{2}\sin^2\theta - 1\right) \mp \\ \mp \frac{1}{4\pi}\sqrt{2\sigma_{PS}\,\sigma_{PD}}\cos(\delta_2 - \delta_0)\left(\frac{3}{2}\sin^2\theta - 1\right) \tag{A6}$$

Formula (A. 6), certainly, can also be used for finding the differential and total cross section when the Coulomb interaction is taken into account.

Appendix B

We proceed in a manner completely analogous to that employed by Guth and Mullin (Guth 1949):

$$\sigma_{PS} = 4\pi^2 \frac{1}{\hbar c}\hbar\omega < \left|\frac{2}{7} e z \left(0, \frac{1}{2}, m \mid 1, \frac{3}{2}, m\right)\right|^2 > \\ \sigma_{PD} = 4\pi^2 \frac{1}{\hbar c}\hbar\omega \left\{ < \left|\frac{2}{7} e z \left(2, \frac{5}{2}, m \mid 1, \frac{3}{2}, m\right)\right|^2 > + \right. \tag{B1} \\ \left. + < \left|\frac{2}{7} e z \left(2, \frac{3}{2}, m \mid 1, \frac{3}{2}, m\right)\right|^2 > \right\}$$

We take the matrix element $\frac{2}{7} ez$, since this is the dipole moment of the triton —

[1] „—“ or „+“ must be choosen according to sign R_{PS} = sign R_{PD} or not.

α particle system. $< >$ represents the mean value taken over all possible initial states. From Eq. (B. 1) we obtain

$$\sigma_{PS} = 4\pi^2 \frac{e^2}{\hbar c} \hbar\omega \frac{4}{49} \times \frac{1}{9} \times R_{PS}^2$$
$$\sigma_{PD} = 4\pi^2 \frac{e^2}{\hbar c} \hbar\omega \frac{4}{49} \times \frac{10}{45} R_{PD}^2 \tag{B2}$$

from which we arrive at the result

$$\frac{\sigma_{PD}}{\sigma_{PS}} = 2 \frac{R_{PD}^2}{R_{PS}^2} \tag{B3}$$

В. Чиж. *О простой модели ядра ^7Li и её применении к изучению реакции $^7Li(\gamma,^3H)^4He$.*

Для вычисления полного интегрального и дифференциального поперечного сечения реакции $^7Li(\gamma,^3H)^4He$ применена „двучастичная" модель ядра ^7Li. В этой модели принимается, что ядро ^7Li можно считать системой, состоящей из тритона и частицы α, вращающихся вокруг общего центра массы. Полученное поперечное сечение согласуется в основных чертах с сечением экспериментально намеченным, если принять что силы, существующие между тритоном и частицей α, являются силами отталкивающими в состоянии D.

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LETTERS TO THE EDITOR

APPLICATION OF THE HERSCHEL EFFECT FOR SELECTIVE ERASING OF BACKGROUND IN NUCLEAR EMULSION

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Nuclear emulsions are now widely used in nuclear physics. Often there arises the necessity to give the emulsion a heavy exposure to the radiation of a radioactive source or an accelerator. The effect under investigation is then obscured by the background fog produced by γ-rays, creating a serious difficulty in the observation of the proper effect.

The aim of the present work was to improve the visibility of nuclear events in photographic emulsions by the selective erasing of the γ-ray background. This note is confined to the method of the Herschel effect only.

Goldstein and Sherman (1952) investigated the effect upon nuclear emulsion of irradiations by infra-red light. They have found a positive Herschel effect (diminution of blackening) on the background produced by visible light, X-rays, γ-rays, and β-rays in the Eastman NTA emulsion. Tracks of ^{210}Po α-particles were examined only qualitatively and no effect was observed on them.

In the process of erasing of the background one had to expect a perceivable influence of the infra-red light on the density of tracks of low ionizing nuclear particles. We tried to approach to the problem of the improvement of the visibility of events considerating the Herschel effect on nuclear tracks. The signal-to-noise ratio η was chosen to measure the visibility in each particular case. This ratio is defined as follows: $\eta = \frac{S^2}{T}$, where S is the number of grains per unit of length of a track parallel to the plane of emulsion and T is the number of background grains in the volume unit of the emulsion. It can be shown that η is a proper measure of visibility and does not depend on magnification, when the depth of the focus of the microscope is constant.

The measurements were performed on lithium loaded $M2$ emulsion, 120 μ thick, manufactured by W. Markocki (the chair of Phototechnics of the Wrocław Institute

of Technology). The emulsion under investigation was exposed to a beam of slow neutrons and a strong dose of γ-rays of a radium source filtered through 5 cm of Pb. The dose of γ-rays amounted to about 2×10^{11} photons per square cm of emulsion. Afterwards the emulsion was irradiated by the light from a 300 W incandescent lamp filtered through an infra-red filter (Dr B. Lange), the distance between the filament and the emulsion being about 7 cm. In order to get rid of temperature effects, the whole arrangement was cooled in an air stream. Then the emulsion was developed in an amidol developer containing boric acid ($pH = 6.4$) diluted 1 : 4 for 16 min. in the temperature of 20°C and fixed.

The examination of emulsion was performed with a 90 x oil immersion objective. The number of grains of the $\alpha + T$-tracks produced in the reaction $^6\mathrm{Li}\ (n, \alpha)^3\mathrm{H}$ and the number of the background grains contained in a fixed volume of emulsion were counted. The statistical error was kept below 5%.

Fig. 1 shows the number of grains on $\alpha + T$-tracks and the number of grains in the volume of $1000\mu^3$ of the emulsion plotted versus the time of irradiation by infra-red light. The rapid fall of the background can be easily seen on the plot B, the reduction of the grain density on $\alpha + T$-tracks (A) being however considerably less rapid with the increase of time of erasing.

The signal-to-noise ratio was calculated from the curves on fig. 1. Assuming for the non-erased emulsion $\eta = \eta_0$, the curve A on fig. 2 gives for $\alpha + T$-tracks the dependence of signal-to-noise ratio on the time of exposure to the infra-red light.

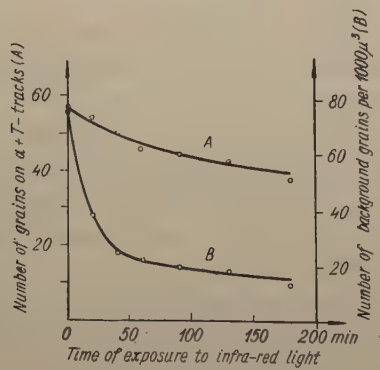

Fig. 1

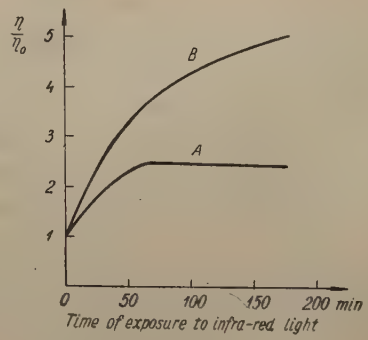

Fig. 2

Similar experiments were performed for α-particle tracks. No change within statistical error of the number of grains on the tracks of ThC′ α-particles could be observed even after a 180 min. exposure to the infra-red light. Curve B on fig. 2 represents the change of η (after an appropriate normalization) for ThC′ α-particles with the increasing time of erasure.

The $M2$ emulsion has a high level of the chemical fog. A considerable improvement of η is expected in emulsions with a lower level of chemical fog.

The analysis of the curves on fig. 2 shows that:

1. The Herschel effect causes an improvement of the visibility of tracks obscured by a strong γ-ray background.

2. There exists an optimum dose of the infra-red light depending on the kind of the observed particle.

3. The value of η/η_0 corresponding to the optimum erasure is varying with the ionizing power of particles.

The authors wish to express their thanks to Prof. Dr H. Niewodniczański for inspiration to this work and many helpful discussions.

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AN IMMERSION BETA-RAY G.M. COUNTER

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In many applications of radioactivity, for example determination of potassium content of solutions, certain processes involving radioactive tracers, etc. thin-walled immersion counters are employed.

As metal may be corroded by the solutions used, immersion counters with an exposed metal cathode have obvious disadvantages. Therefore internally coated, thin-walled glass counters are normally employed.

Internal coating, either by the usual chemical method or by the evaporation technique (1), although technically not a difficult task, nevertheless is time consuming and complicates the construction. Moreover, internally coated counters often show a marked photosensitivity.

In order to detect or make qualitative measurements of low level β activity of conductive liquids, the author has made a very simple modification of the immersion glass counter. The counting tube has neither an internal or an external metal coating; it acts as a Maze — type counter, where the very liquid under investigation plays the role of the external aquadag coating in the original Maze counter.

Fig. 1. shows how one of the simplest forms of this type of counters is employed.

The tubes are made of ordinary soda glass and are reduced in thikness to a desired degree either by ordinary glass-blowing or by the chemical procedure described by Elliot and Lee (2); they are filled by the standard mixture.

Radiation from a conductive solution may be detected and measured by merely lowering the counter into the solution contained in a cylindrical metal vessel. The vessel should be connected electrically to the negative terminal of the H. T. supply.

In quantitative measurements, the free surface of the solution must be kept at a definite level with respect to the counting tube (a few milimeters above the upper boundary of the thin wall of the tube).

The background of the counter may be determined when the metal vessel in fig. 1 is filled with ordinary tap water.

To reduce the background, the entire arrangement of Fig. 1 must be enclosed in a lead shield.

The counters under discussion were used by the author for quantitative determinations of potassium in potassium salts according to the method of Barnes and Salley (3).

Fig. 1

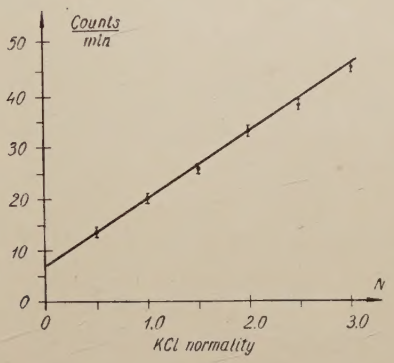

Fig. 2

Fig. 2. shows the calibration curve of one of the counters used in these determinations.

It is obvious, that the constructional details of a counter based on the principle described here, may be worked out in a variety of ways.

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